Theory of Individual and Strategic Decisions

Course manual

Course manual EBC4197
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Department of Microeconomics and Public Economics
Maastricht University
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Contents

1 Introduction

1.1 Aim of the course

The aim of the course is to provide an overview of the standard analytical tools that are used for modelling decision making. Decision Theory focuses on individual decision problems, and the general purpose is twofold. On the one hand, it provides us with a framework within which common choice patterns and individual characteristics can be identified and studied. For instance, it allows us to formally define and measure risk attitudes. On the other hand, it provides us with a natural benchmark to study Game Theory, which in turn focuses on decisions in environments with strategic interactions among several agents. The course mainly focuses on rational agents with unlimited reasoning abilities. These agents are used as a benchmark in most economic models.

1.2 Prerequisites

It is expected that the students who take this course have a minimum level of familiarity with undergraduate mathematics and statistics. As an indication, students are expected to be familiar with the concepts discussed in Chapter 19 (Mathematical Appendix) of *Game Theory: An Introduction* by Steven Tadelis, and in the online Appendix A of *Rational Choice* by Itzhak Gilboa. Although no prior knowledge in microeconomic theory or game theory is necessary, as the course is self-contained, it is recommended (especially for students with no background in economics) to read Chapters 1–3 of *Economics* by Daron Acemoglu, David Laibson and John List, in order to get an idea of the main purposes and the methological underpinnings of economics research.

1.3 Expectations

Though the course is demanding, it is structured in a way such that a student can be successful by regularly attending the scheduled meetings, and studying according to the recommendations below. A rough estimate of the *self-study time* needed for the course is approximately 20 hours per week (for an average student to master the material at an average level). In order to pass the course, the students will need to complete certain tasks (final exam, participation, problem sets, self-evaluation). Some of these tasks will be undertaken in working groups that will be formed at the beginning of the first lecture. Each working group will coincide with the subgroups that the scheduling office has formed.

2 Course structure

2.1 Description

The course consists of 14 meetings (7 theory meetings and 7 exercise tutorials). Lectures have been already recorded in videos that will be provided to the students at the beginning of the course, together with the corresponding notes.

- **Theory meetings:** These take place once a week, every Monday. Before each theory meeting, you are expected to have watched the corresponding lectures and to have read the corresponding literature. During the meeting, you can ask questions on points that you do not understand. In order to make use of these meeting in an optimal way, you should not show up unprepared. Participation in these meetings is not formally mandatory, but nevertheless strongly advised.

  All theory meetings will take place online.

- **Exercise tutorials:** These take place once a week, every Thursday. Before each tutorial you are expected to have worked on the designated problem set, together with the fellow students from your subgroup. The answers should be typewritten and sent by midnight on Wednesday before the session to ebc4197@gmail.com. Solutions must be submitted in working groups. During the tutorial, you will spend some time with the tutor to solve these problems together. Subsequently, the subgroup is expected to go back to their own solutions and grade them. You may be asked to also grade (anonymous) solutions of other working groups. Participation in the tutorials is mandatory.

  Some tutorials will take place online, according to the schedule.

2.2   Schedule

The precise schedule of the course is as follows:

1. Theory meeting on **preferences and utility**

   Instructor: Elias Tsakas

   Literature: Appendix A; Osborne & Rubinstein (Ch. 1); Bonanno (Decision Making; Ch. 1,2)

   Videos: 1–8

2. Tutorial on **preferences and utility**

   Instructor: Eveline Vandewal/Toygar Kerman

   Problems: A.1 and A.2

3. Theory meeting on **choice theory**

   Instructor: Elias Tsakas

   Literature: Appendix B; Osborne & Rubinstein (Ch. 2)

   Videos: 9–15

4. Tutorial on **choice theory**

   Instructor: Eveline Vandewal/Toygar Kerman

   Problems: B.1 and B.2

5. Theory meeting on **decision theory under uncertainty**

   Instructor: Elias Tsakas

   Literature: Appendix C; Osborne & Rubinstein (Ch. 3); Bonanno (Decision Making; Ch. 3,5)

   Videos: 16–28

6. Tutorial on **decision theory under uncertainty**

   Instructor: Eveline Vandewal/Toygar Kerman

   Problems: C.1 and C.2

7. Theory meeting on **ordinal strategic-form games**

   Instructor: Elias Tsakas

   Literature: Appendix D.1; Bonanno (Game Theory; Ch. 1,2); Osborne & Rubinstein (Ch. 15)

   Videos: 29–34

8. Tutorial on **on ordinal strategic-form games**

   Instructor: Eveline Vandewal/Toygar Kerman

   Problems: D.1 and D.2

9. Theory meeting on **cardinal strategic-form games**

   Instructor: Elias Tsakas

   Literature: Appendix D.2; Bonanno (Game Theory; Ch. 6); Osborne & Rubinstein (Ch. 15)

   Videos: 35–39

10. Tutorial on **cardinal strategic-form games**

    Instructor: Eveline Vandewal/Toygar Kerman

    Problems: D.3 and D.4

11. Theory meeting on **on extensive-form games with perfect information**

    Instructor: Elias Tsakas

    Literature: Appendix E.1; Bonanno (Game Theory; Ch. 3), Osborne & Rubinstein (Ch. 16)

    Videos: 40–44

12. Tutorial on **extensive-form games with perfect information**

    Instructor: Eveline Vandewal/Toygar Kerman

    Problems: E.1 and E.2

13. Theory meeting on **extensive-form games with imperfect information**

    Instructor: Elias Tsakas

    Literature: Appendix E.2; Bonanno (Game Theory; Ch. 4), Osborne & Rubinstein (Ch. 16)

    Videos: 45–46

14. Tutorial on **extensive-form games with imperfect information**

    Instructor: Eveline Vandewal/Toygar Kerman

    Problems: E.3 and E.4

2.3 Literature

Throughout the course we will be using various textbooks. For the theory sessions the students are expected to have read the corresponding chapters.

- MAIN TEXTBOOKS (used throughout the course):

  - BONANNO, G. (2015). *Game Theory*. Open access textbook (free download).
  - BONANNO, G. (2018). *Decision Making*. Open access textbook (free download).
  - OSBORNE, M. & RUBINSTEIN, A. (2020). *Models in Microeconomic Theory*. Open access textbook (free download).

- ADDITIONAL TEXTBOOKS (recommended for further reading):

  - GILBOA, I. (2009). *Theory of Decision under Uncertainty*. Econometric Society Monographs.
  - RUBINSTEIN, A. (2012). *Lecture Notes in Microeconomic Theory: The Economic Agent*. Princeton University Press (free download).
  - TADELIS, S. (2013). *Game Theory: An Introduction*. Princeton University Press.

3 Performance assessment

The final grade will be calculated based on the performance in the following tasks (with the corresponding weights in parenthesis):

- **Final exam (72%):** The final exam will be an oral exam, consisting of both theoretical questions and problems. You will receive more questions on the exact format of the exam.

- **Problem sets (21%):** There are 7 problem sets, one for each tutorial meeting. Your answers to the problem sets should be typewritten in a clear and professional way. Handwritten answers will not be accepted. The answers should be sent in pdf format to ebc4197@gmail.com by midnight before the corresponding exercise tutorial (with the name of the tutor on the subject line of the email and the names of all the group members on the front page of your answer sheet). The deadline is strict and late submissions will not be accepted. The answers should be submitted in working groups. Individual answers will not be accepted. If a member of the group has zero contribution to solving a problem set, it should be reported to the course coordinator.

- **Self-evaluation (7%):** Using the answers discussed in class with the tutor, you must go back to your submitted answers (i.e., the ones you sent by midnight the day before) and grade your own problem set on a scale $0 - 10$. Your self-evaluation will be compared to the grade that your tutor gives you (which will also be on a scale $0 - 10$), and if the two are at most 1 point apart from each other, then you will receive a bonus of 1% of the full score to your final grade. Note that in order to be eligible for the bonus, your actual grade (i.e., the one you receive from your tutor) must be at least $5/10$. It is always possible that you are asked to justify your self-evaluation, by means of explaining why you graded your problem set the way you did. Failure to do so, may result in loosing your bonus. You may furthermore be asked to evaluate another subgroup's problem set on top of your own. In this case, a double-blind procedure will be followed.

- **Participation (pass/fail):** You are expected to be present (physically and mentally) in at least 5 of the 7 tutorials. If you miss more than 2 tutorials, you fail the course. In extenuating circumstances, you may be allowed to compensate more than 2 absences with some additional assignment. Whether this will be the case, and which the extra assignment will be, is entirely at the discretion of the course coordinator.

The final grade will be on a scale $1 - 10$, rounded to the nearest half point. To pass the course one needs at least 55 points. That is, 53 points and 54 points are downgraded to 5 and therefore do not suffice to pass the course. For the students who fail the course, there will be a resit exam. The students who take the resit carry with them the grades received for the other tasks. Note that partial grades obtained in this course remain valid for a period of three years, i.e., the academic year in which the grades were obtained plus two subsequent academic years.

4   Contact information

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Part I
Decision Theory

A   Preferences and Utility

OUTCOMES: We begin with an arbitrary set $X \subseteq \mathbb{R}^n$ of (sure) outcomes. For instance, outcomes can be thought as monetary outcomes (in which case $X = \mathbb{R}_+$) or as consumption bundles (in which case $X = \mathbb{R}_+^n$ where n is the number of goods in the bundle). The set of outcomes is assumed to be known both to the decision maker and to the economist.

PREFERENCES: The decision maker has preferences over the outcomes. Importantly, these preferences are in principle known only to the decision maker and not to the economist. Preferences are formally modelled with a binary relation $\succeq \subseteq X \times X$ over pairs of outcomes. We write $x \succeq y$ whenever $(x, y) \in \succeq$. Intuitively, think of a decision maker who takes every single pair of outcomes (x, y) and asks himself "do I find x to be at least as good as y". Obviously the answer will be positive for some pairs and negative for some other pairs. Then, $\succeq$ is the subset of pairs for which the answer to the previous question was positive.

STRICT PREFERENCE: a binary relation $\succ \subseteq X \times X$ defined by $x \succ y$ if and only if $x \succeq y$ and $y \not\succeq x$. The interpretation is that the individual finds x to be strictly better than y.

INDIFFERENCE: a binary relation $\sim \subseteq X \times X$ defined by $x \sim y$ if and only if $x \succeq y$ and $y \succeq x$.

Question A.1. *Explain the following relations: (a) $\succ \subseteq \succeq$, (b) $\sim \subseteq \succeq$, (c) $\succ \cup \sim = \succeq$.*

AXIOMS: they are natural assumptions we impose on the preferences of the individual. They provide structure (i.e., internal consistency) to the preferences, which is needed in order to be able to build a systematic theory and form testable hypotheses. From now onwards we assume the following axioms:

(A_1) COMPLETENESS: for all $x, y \in X$, it is the case that $x \succeq y$ or $y \succeq x$.

(A_2) TRANSITIVITY: for all $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$ then $x \succeq z$.

Problem A.1. *Prove that a preference relation $\succeq$ over X satisfies completeness and transitivity if and only if the corresponding strict preference relation $\succ$ satisfies*

(A_1') *Asymmetry: For any two outcomes $x, y \in X$ it cannot be the case that $x \succ y$ and $y \succ x$ hold simultaneously.*

(A_2') *Negative transitivity: For every three outcomes $x, y, z \in X$ with $x \succ y$, it is the case that $x \succ z$ or $z \succ y$.*

Question A.2 (Challenging)**.** *Prove the following statement and provide some intuition: if X is a finite set and $\succeq$ satisfies completeness and transitivity, then there exists a most-preferred and a least-preferred outcome (not necessarily unique).*

UTILITY REPRESENTATION OF $\succeq$: a function $u : X \to \mathbb{R}$ such that, for all $x, y \in X$

$$x \succeq y \Leftrightarrow u(x) \geq u(y).$$

Intuitively a utility representation is an (unambiguous) translation from the language of the preferences to the language of mathematics.

Question A.3. *Prove the following statement and provide some intuition: if $u : X \to \mathbb{R}$ is a utility representation of $\succeq$ and $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then the function $v : X \to \mathbb{R}$ which is defined by $v(x) = f(u(x))$ is also a utility representation of $\succeq$.*

Problem A.2. *Let $\succeq$ be a complete and transitive preference relation on X. Then, prove that $u : X \to \mathbb{R}$ is a utility representation of $\succeq$ if and only if the following two conditions hold:*

(i) For all $x, y \in X$ with $x \succ y$, it is the case that $u(x) > u(y)$.

(ii) For all $x, y \in X$ with $x \sim y$, it is the case that $u(x) = u(y)$.

PROPERTIES OF UTILITY: each utility function has some (mathematical) properties, e.g., increasing, convex, continuous, etc. As there are multiple utility functions representing the same preferences (Question A.3), different utility functions may satisfy different properties. Utility functions are characterized by its relevant properties, i.e, those properties that we actually use to draw some conclusions about the decision maker's behavior. For instance, if the utility function is convex but we never (explicitly or implicitly) use convexity to make any statement about this decision maker's behavior, then convexity is not a relevant property. We classify properties in two types:

- ORDINAL UTILITY: Properties shared by all utility representations of $\succeq$ are called ordinal. In this sense, whether an ordinal property is satisfied by some utility representation of $\succeq$ depends only on the preferences. We say that a utility function is ordinal if we only use ordinal properties. Ordinal utilities do not have any meaning and are usually employed when X is discrete (e.g., when $X = \mathbb{N}$).

- CARDINAL UTILITY: Properties satisfied only by some utility representations of $\succeq$ are called cardinal properties. In this sense, whether a cardinal property is satisfied by a utility function or not does not depend only on the preferences. We say that a utility function is cardinal if we also use cardinal properties. Cardinal utilities typically carry some meaning and are often employed when X is a continuum (e.g., when $X = \mathbb{R}_+$)

Question A.4. *Take the functions $u(x) = \ln(1 + x)$ and $v(x) = x^2$. Verify that both are representations of the same preferences over $\mathbb{R}_+$, and that these preferences satisfy $(A_1) - (A_2)$. Relate your answer to the one of Question A.3. Is the property "the utility function is strictly increasing with respect to money" ordinal or cardinal? How about the property "the utility function is convex with respect to money"? Explain.*

(A_0) MONOTONICITY: for all $x = (x_1, \ldots, x_n) \in X$ and $y = (y_1, \ldots, y_n) \in X$, if $x_k \geq y_k$ for all $k = 1, \ldots, n$ then $x \succeq y$.

Question A.5. *Suppose that the preference relation $\succeq$ is represented by the utility function $u : X \to \mathbb{R}$. Then, prove the following statement and provide some intuition: $\succeq$ satisfies monotonicity if and only if u is an increasing function.*

Question A.6 (Challenging). *Prove and provide intuition for the following statement: A property satisfied by $u : X \to \mathbb{R}$ is ordinal if and only if it is satisfied by $v : X \to \mathbb{R}$ where $v(x) = f(u(x))$ for arbitrary strictly increasing function $f : \mathbb{R} \to \mathbb{R}$.*

Question A.7. *Take the function $u(x) = 20,000x - x^2$. Is this a utility representation of some preferences over $\mathbb{R}_+$ that satisfy $(A_1)-(A_2)$? If yes, do these preferences satisfy (A_0)? Draw the graph of u and describe in words a decision maker with this utility function. In your intuitive description, which statements that you used correspond to ordinal properties and which ones to cardinal properties? Now, provide an alternative utility function that represents the same preferences, but has different some of the cardinal properties that you have identified.*

Theorem A.1. *If X is a finite set and $\succeq$ satisfies completeness and transitivity, then there exists a utility representation.*

Question A.8. *Prove the previous theorem and provide some intuition.*

Example A.1. (LEXICOGRAPHIC PREFERENCES). Take $X = \mathbb{R}_+^2$ to be the set of all (non-negative) consumption bundles of two goods. Let the preference relation $\succeq$ be defined as follows: For each pair of alternatives $x = (x_1, x_2)$ and $y = (y_1, y_2)$ the agent first compares the first coordinate and prefers the alternative with the highest value, completely disregarding the second coordinates. The agent will use the second coordinates only to break the ties resulted when comparing the first coordinates. In other words, the agent cares infinitely more about the first coordinate than the second one. Formally,

$$x \succeq y \Leftrightarrow \text{ either } x_1 > y_1 \text{ or simultaneously } x_1 = y_1 \text{ and } x_2 \geq y_2. \tag{A.1}$$

We call these preferences lexicographic. ◁

Question A.9. *Verify that the lexicographic preferences of the previous example satisfy completeness, transitivity and monotonicity.*

Question A.10 (Really challenging). *Explain the following statement (by means of an intuitive argument): The lexicographic preferences of the previous example do not have a utility representation.*

 Hint: it is not possible to fit uncountably many disjoint intervals in $\mathbb{R}$. Formally, suppose that to every number $z \in \mathbb{R}_+$ you associate some interval $[a_z, b_z]$. Then, it will necessarily be the case that you can find $z, z' \in \mathbb{R}_+$ such that $[a_z, b_z] \cap [a_{z'}, b_{z'}] \neq \emptyset$.

B   Choice theory

CHOICE PROBLEM: the set of available choices (sometimes called actions). Each choice yields one outcome (with certainty), which is why we identify each choice problem with some nonempty $A \subseteq X$.[1] In this sense, two choices that induce the same outcome, essentially constitute the same choice. We denote the set of all choice problems by $\mathcal{A}$.

RATIONAL CHOICE: a choice $a \in A$ such that $a \succeq b$ for all $b \in A$. Obviously, whether a choice is rational or not depends on both what is available (i.e., the set A) and what is preferred (i.e., the relation $\succeq$). If the preference relation is represented by a utility function u, then a choice is rational if it maximizes u in A, i.e., formally if it is one of the choices in $\{a \in A : u(a) \geq u(b) \text{ for all } b \in A\}$.

Question B.1. *Prove the following statement and provide intuition: if X is finite and $\succeq$ satisfies completeness and transitivity, then there exists an optimal choice in every choice problem $A \in \mathcal{A}$.*

Question B.2. *Suppose that the function $v(x) = x^2$ represents $\succeq$ over $\mathbb{R}_+$. Provide an example of some choice problem that does not have an optimal choice.*

AIM: The economist does not know the decision maker's preferences, and instead makes some assumptions (in the form of axioms) about how these look like. These assumptions should be tested.

EXPERIMENT: a sequence of choice problems $\mathcal{E} \subseteq \mathcal{A}$. An experiment is complete if it contains all choice problems (i.e., $\mathcal{E} = \mathcal{A}$), and it is incomplete otherwise (i.e., $\mathcal{E} \subsetneq \mathcal{A}$).

DATASETS: data may come in two forms. In particular, the economist observes all the choices that the decision maker picks from each choice problem $A \in \mathcal{E}$ (e.g., we have many observations from each choice problem) or the economist observes only one choice from each choice problem $A \in \mathcal{E}$ (e.g., we have one observation from each choice problem). Then, accordingly we define the following:

- CHOICE CORRESPONDENCE: $C(A) \subseteq A$ for each $A \in \mathcal{E}$

- CHOICE FUNCTION: $c(A) \in A$ for each $A \in \mathcal{E}$

Obviously, a choice correspondence contains more information than a choice function. In fact, you can take $c(A) \in C(A)$, i.e., $c(A)$ is one of the choices that the decision maker makes when facing A.

RATIONALIZING CHOICE: Irrespective of what type of data we have in our hands (i.e., whether the data set is complete or not, and whether we observe a choice function or a choice correspondence), we would like to use this data to test our decision-theoretic model. Formally, we are asking:

> *Is there a preference relation $\succeq$ satisfying the axioms we have assumed, which is consistent with the choices that we observe in our dataset?*

If we cannot find such a preference relation, then we reject the hypothesis that this model describes the decision-maker's preferences (and a fortiori her behavior). On the other hand, if there is a such a relation, then we cannot reject our model, and we can continue operating under the assumption that the decision maker's behavior is driven by these preferences. It is very important to stress that we are not claiming that these are indeed the decision maker's preferences, but rather that her observed behavior is consistent with such preferences. Perhaps, if we get more data, we will then reject this model, but for the time being, this model still does a good job. This is called the *"as if approach"* in economics, and the methodology that we will use to test our models agains the data is called *revealed preference*.

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[1]Throughout this section, unless explicitly stated otherwise, we assume that X is finite.

B.1 Choice correspondence

RATIONALIZING A CHOICE CORRESPONDENCE: Given an experiment $\mathcal{E}$, the choice correspondence C is rationalized by the preference relation $\succeq$, if for each $A \in \mathcal{E}$,

$$C(A) = \{a \in A : a \succeq b \text{ for all } b \in A\}.$$

In other words, $\succeq$ rationalizes C if $C(A)$ is the set of rational choices from A given $\succeq$.

DIRECT REVEALED PREFERENCE: given the choice correspondence C, choice a is said to be directly revealed-preferred to b whenever

there exists some $A \in \mathcal{E}$ with both $a, b \in A$, such that $a \in C(A)$.

In this case we write $a \succeq_C b$. The idea is that a is observed to be chosen while b is also available. Crucially, we need to remember that $\succeq_C$ is not necessarily the same as the decision maker's true preferences. It is only an inference we make about the true preferences, based on the data.

(DIRECT) STRICT REVEALED PREFERENCE: given the choice correspondence C, choice a is said to be strictly revealed-preferred to b whenever

there exists some $A \in \mathcal{E}$ with both $a, b \in A$, such that $a \in C(A)$ and $b \notin C(A)$.

In this case we write $a \succ_C b$. The idea is that while both a and b are available, only a is chosen.

Question B.3. *Provide an example of an experiment $\mathcal{E}$ and a choice correspondence C such that $a \succ_C b$ and $b \succ_C a$.*

WEAK AXIOM OF REVEALED PREFERENCE (WARP): for every $A, B \in \mathcal{E}$ with $a, b \in A \cap B$,

if $a \in C(A)$ and $b \in C(B)$, then $b \in C(A)$ and $a \in C(B)$.

Question B.4. *Write formally the following statement: WARP is violated.*

Problem B.1. *Prove that the choice correspondence C satisfies WARP if and only if it is not the case that $a \succ_C b$ and $b \succeq_C a$ for any $a, b \in X$.*

Theorem B.1. *For an experiment $\mathcal{E}$ and a choice correspondence C, the following hold:*

(i) If C violates WARP, then C cannot be rationalized by any complete and transitive $\succeq$.

(ii) If C satisfies WARP and $\mathcal{E}$ is complete, then C can be rationalized by a (unique) complete and transitive $\succeq$.

Question B.5. *Take $X = \{a, b, c\}$ and $\mathcal{E} = \{\{a, b\}, \{b, c\}, \{a, c\}\}$. Prove that the choice correspondence $C(\{a, b\}) = \{a\}$, $C(\{b, c\}) = \{b\}$ and $C(\{a, c\}) = \{c\}$ satisfies WARP. Moreover, prove that C cannot be rationalized by any complete and transitive $\succeq$.*

Question B.6. *If we enrich the experiment of Question B.5 by also adding $\{a, b, c\}$ to $\mathcal{E}$, is there some $C(\{a, b, c\})$ such that the choice correspondence still satisfies WARP?*

Question B.7 (Challenging). *Prove that if the choice correspondence C satisfies WARP, then for every $A, B \in \mathcal{E}$ with $B \subseteq A$ it also satisfies the following:*

(i) (SEN'S) PROPERTY α : *if* $a \in B$ *and* $a \in C(A)$, *then* $a \in C(B)$,

(ii) (SEN'S) PROPERTY β : *if* $a, b \in C(B)$ *and* $a \in C(A)$, *then* $b \in C(A)$.

Moreover, prove that the converse is true for a complete experiment $\mathcal{E}$, *i.e., if properties* α *and* β *hold, then so does WARP.*

Question B.8. *Using Theorem B.1 and Question B.7, prove the following statements:*

(i) If C *violates violates property* α *or property* β, *then* C *cannot be rationalized by any complete and transitive* $\succeq$.

(ii) If C *satisfies both properties* α *and* β *in a complete experiment* $\mathcal{E}$, *then* C *can be rationalized by a (unique) complete and transitive* $\succeq$.

INDIRECT REVEALED PREFERENCE: given the choice correspondence C, choice a is said to be indirectly revealed-preferred to b whenever

there exists a sequence $\{a_1, \ldots, a_N\}$ in X, such that $a \succeq_C a_1 \succeq_C \cdots \succeq_C a_N \succeq_C b$.

In this case we write $a \succeq_C^* b$. The idea is that even if a is not observed to be preferred to b, it is observed to be preferred to some other choice, which is preferred to some other choice, and so on, which is preferred to b.

GENERALIZED AXIOM OF REVEALED PREFERENCE (GARP): for every $a, b \in X$,

it is not the case that $a \succeq_C^* b$ and $b \succ_C a$.

The intuition is that it cannot be the case that revealed preference about two choices is conflicting. The only difference is that WARP postulates that direct revealed preferences are not conflicting (Problem B.1), whereas GARP generalizes this requirement to indirect revealed preference too.

Theorem B.2. *For an experiment* $\mathcal{E}$ *and a choice correspondence* C, *the following hold:*

(i) If C *violates GARP, then* C *cannot be rationalized by any complete and transitive* $\succeq$.

(ii) If C *satisfies GARP, then* C *can be rationalized by some complete and transitive* $\succeq$.

Question B.9. *Prove that the choice correspondence* C *from Question B.5 violates GARP.*

B.2   Choice function

RATIONALIZING A CHOICE FUNCTION: Given an experiment $\mathcal{E}$, the choice function c is rationalized by the preference relation $\succeq$, if for each $A \in \mathcal{E}$,

$$c(A) \in \{a \in A : a \succeq b \text{ for all } b \in A\}.$$

In other words, $\succeq$ rationalizes c if $c(A)$ is one of rational choices from A given $\succeq$.

Question B.10. *Show that every choice function* c *is rationalized by a complete and transitive* $\succeq$.

So, we test completeness and transitivity together with an additional axiom:

(NI) NO-INDIFFERENCE: for any $x, y \in X$, either $x \succ y$ or $y \succ x$.

That is, we ask whether the choice function c can be rationalized by a complete and transitive preference relation $\succeq$ that does not make the decision maker indifferent between any two choices. By the way, such a preference would be called a linear order.

> **Problem B.2.** *Prove that a choice function c is rationalized by a preference relation $\succeq$ that satisfies completeness, transitivity and no-indifference if and only if $\succeq$ rationalizes the choice correspondence C such that $C(A) = \{c(A)\}$ for each $A \in \mathcal{E}$. What is the intuition?*

PROPERTY α (FOR CHOICE FUNCTIONS): for every $A, B \in \mathcal{E}$ with $B \subseteq A$,

$$\text{if } c(A) \in B, \text{ then } c(A) = c(B).$$

Question B.11. *Explain how property α for choice correspondences relates to property α for choice functions.*

Theorem B.3. *For an experiment $\mathcal{E}$ and a choice function c, the following hold:*

   (i) If c violates property α, then c cannot be rationalized by any $\succeq$ that satisfies completeness, transitivity and no-indifference.

   (ii) If c satisfies property α and $\mathcal{E}$ is complete, then c can be rationalized by a (unique) $\succeq$ that satisfies completeness, transitivity and no-indifference.

Question B.12. *Take $X = \{a, b, c\}$ and $\mathcal{E} = \{\{a, b\}, \{b, c\}, \{a, c\}\}$, like in Question B.5. Prove that the choice function $c(\{a, b\}) = \{a\}$, $c(\{b, c\}) = \{b\}$ and $c(\{a, c\}) = \{c\}$ satisfies property α. Moreover, prove that c cannot be rationalized by any $\succeq$ that satisfying completeness, transitivity and no-indifference.*

C   Decision theory under uncertainty

GENERAL AIM: understand how people choose under uncertainty. Interesting questions of economic relevance that we want to answer are for instance: "how can we identify an individual's risk attitudes" or "how do we elicit an individual's (latent) subjective beliefs"?

CHOICE DOMAIN: we use two different domains of alternatives, depending on the type of uncertainty the individual faces. In particular, we use the set of lotteries to study decisions under objective uncertainty (risk) and the set of acts to study decisions under subjective uncertainty. In both cases, the set of outcomes is assumed to be monetary, i.e., $X \subseteq \mathbb{R}$.

Question C.1. *What is the difference between objective uncertainty and subjective uncertainty?*

C.1   Objective uncertainty

LOTTERY: a random experiment p that yields each monetary outcome $x \in X \subseteq \mathbb{R}$ with some known probability $p(x)$. Even when X is infinite, only finitely many outcomes receive positive probability. So, a lottery p is identified by the probabilities it assigns to each outcome, and it is typically denoted by $p = (p(x_1) \times x_1, \ldots, p(x_K) \times x_K)$ where $x_1, \ldots, x_K \in X$ and $p(x_1) + \cdots + p(x_K) = 1$. The set of all lotteries is denoted by $\mathcal{L}(X)$, and this will be our set of alternatives. Whenever the set of outcomes (over which the lotteries are defined) is clear from the context, we omit X and we simply write $\mathcal{L}$. Each outcome $x \in X$ can also be seen as the (degenerate) lottery $(1 \times x)$. In this sense, with slight abuse of notation, with sometimes write $X \subseteq \mathcal{L}$.

Question C.2. *Explain the following statement: A lottery is a probability distribution over $\mathbb{R}$. In this sense, only the underlying probabilities matter and not the actual random experiment. Provide an example of two different random experiments that induce the same lottery.*

PREFERENCES OVER LOTTERIES: complete and transitive preferences $\succeq$ over $\mathcal{L}$. Strict preferences and indifference are defined as above. Since $X \subseteq \mathcal{L}$, preferences over lotteries also induce preferences over sure outcomes, i.e., we obtain $x \succeq y$ if and only if $(1 \times x) \succeq (1 \times y)$. The induced preferences over sure outcomes are (strictly) monotonic, i.e., the decision maker prefers higher monetary payoffs over lower ones.

Question C.3 (Expected payoff). *Consider a decision maker that compares any two lotteries $p = (p(x_1) \times x_1, \ldots, p(x_K) \times x_K)$ and $q = (q(y_1) \times y_1, \ldots, q(y_L) \times y_L)$ on the basis of the expected monetary payoff that they induce, i.e.,*

$$p \succeq q \Leftrightarrow \sum_{k=1}^{K} p(x_k) x_k \geq \sum_{\ell=1}^{L} q(y_\ell) y_\ell.$$

Verify that this preference relation is complete and transitive. Verify that the induced preferences over X are monotone.

Question C.4 (Pesimist). *Consider a decision maker that compares any two lotteries $p = (p(x_1) \times x_1, \ldots, p(x_K) \times x_K)$ and $q = (q(y_1) \times y_1, \ldots, q(y_L) \times y_L)$ on the basis of the worst case scenario, i.e.,*

$$p \succeq q \Leftrightarrow \min\{x_1, \ldots, x_K\} \geq \min\{y_1, \ldots, y_L\}.$$

Verify that this preference relation is complete and transitive. Verify that the induced preferences over X are monotone.

UTILITY REPRESENTATION: the preferences over lotteries are represented by a function $u : \mathcal{L} \to \mathbb{R}$ such that, for every pair of lotteries $p, q \in \mathcal{L}$,

$$p \succeq q \Leftrightarrow u(p) \geq u(q).$$

EXPECTED UTILITY: We are particularly interested in utility functions that can be written as expectations. Formally, this is done by first assigning utilities to the sure outcomes via the function $u : X \to \mathbb{R}$ (called Bernoulli utilities), and then assigning to each lottery $p = (p(x_1) \times x_1, \ldots, p(x_K) \times x_K) \in \mathcal{L}$ the expected utility

$$\mathbb{E}_p(u) = \sum_{k=1}^{K} p(x_k) u(x_k).$$

Obviously, once we have assigned Bernoulli utility to every sure outcome, the expected utility of each lottery follows automatically, and we do not have any flexibility in adjusting it. Then, the question is whether we can always find Bernoulli utilities in a way such that for every $p, q \in \mathcal{L}$,

$$p \succeq q \Leftrightarrow \mathbb{E}_p(u) \geq \mathbb{E}_q(u).$$

If this is indeed possible, we say that the preferences $\succeq$ have a vNM expected utility representation.

Question C.5. *Do the preferences in Questions C.3 and C.4 have an expected utility representation? If yes, identify appropriate Bernoulli utilities.*

FIRST-ORDER STOCHASTIC DOMINANCE: Lotteries $p = (p(x_1) \times x_1, \ldots, p(x_K) \times x_K)$ first-order stochastically dominates lottery $q = (q(y_1) \times y_1, \ldots, q(y_L) \times y_L)$, if for every payoff $\alpha \in \mathbb{R}$,

$$\sum_{x_k \geq \alpha} p(x_k) \geq \sum_{y_\ell \geq \alpha} q(y_\ell).$$

That is, for any $\alpha \in \mathbb{R}$, the probability of receiving at least α is greater under p than it is under q.

Theorem C.1 (First-order stochastic dominance). *The lottery p first-order stochastically dominates the lottery q if and only if $\mathbb{E}_p(u) \geq \mathbb{E}_q(u)$ for all (weakly increasing) Bernoulli utility functions.*

Question C.6 (Very challenging). *Prove the previous theorem. Then, explain the following statement: not all lotteries can be compared with each other on the basis of FOSD (i.e., formally speaking FOSD is an incomplete relation).*

vNM AXIOMS: There are four simple axioms that guarantee the existence of a vNM EU representation. These are completeness and transitivity that we have already discussed, plus two additional ones, (A_3) continuity and (A_4) independence.

Theorem C.2 (vNM EU Theorem). *The preferences $\succeq$ over $\mathcal{L}$ have a vNM expected utility representation if and only if they satisfy completeness, transitivity, continuity and independence.*

Question C.7. *Assume that the preferences $\succeq$ over $\mathcal{L}$ have a vNM EU representation. Then, prove formally and provide intuition for the following statement: If $x > y$ and $\alpha > \beta$, then $\big(\alpha \times x, (1 - \alpha) \times y\big) \succ \big(\beta \times x, (1 - \beta) \times y\big)$.*

Problem C.1. *Assume that the preferences $\succeq$ over $\mathcal{L}$ have a vNM EU representation, and consider five monetary payoffs $x_1 > x_2 > x_3 > x_4 > x_5$. Prove formally and provide intuition for the following statements:*

(a) There exist real numbers $\alpha \in (0,1)$ and $\beta \in (0,1)$ such that

$$\big(\alpha \times x_1, (1-\alpha) \times x_5\big) \sim \big(\beta \times x_2, (1-\beta) \times x_4\big).$$

(b) For any two real numbers $\alpha \in (0,1)$ and $\beta \in (0,1)$ it is the case that

$$\big(\alpha \times x_1, (1-\alpha) \times x_2\big) \succeq \big(\beta \times x_3, (1-\beta) \times x_2\big).$$

Question C.8. *Prove formally and provide intuition for the following statement: Suppose that $u : X \to \mathbb{R}$ is a Bernoulli utility function. Then, $v : X \to \mathbb{R}$ is also a Bernoulli utility function if and only if $v(x) = \alpha + \beta u(x)$ for all $x \in X$, where $\alpha \in \mathbb{R}$ and $\beta > 0$.*

Question C.9 (Challenging). *Explain the following statement: the property that "the utility function has an expected utility form" is cardinal, i.e., there are also utility representations of $\succeq$ that cannot be written as an expectation. Provide an example.*

RISK ATTITUDES: Why are we so keen in using expected utility representations, even though there are other utility functions representing the same vNM preferences? One advantage of expected utility (besides its intuitive appeal) is that it allows us to identify risk attitudes. Let us first provide a following classification of decision makers, depending on how they treat risk. We first take an arbitrary sure outcome $x \in X$ and a lottery $p = (p(x_1) \times x_1, \ldots, p(x_K) \times x_K) \in \mathcal{L}$ with the same expected monetary payoff, i.e.,

$$x = \sum_{k=1}^{K} p(x_k) x_k.$$

Then, we consider the following three types:

- RISK AVERSE: a decision maker who prefers the sure outcome, i.e., $x \succeq p$. The decision maker is strictly risk averse if for all such pairs of sure outcomes and (expected-payoff-equivalent) lotteries the preference is strict, i.e., $x \succ p$.

- RISK SEEKING: a decision maker who prefers the lottery, i.e., $p \succeq x$. The decision maker is strictly risk seeking if for all such pairs of sure outcomes and (expected-payoff-equivalent) lotteries the preference is strict, i.e., $p \succ x$.

- RISK NEUTRAL: a decision maker who is indifferent between the two, i.e., $x \sim p$.

Now, expected utility representations are useful, because they allow us to classify decision makers depending on their risk attitudes.

Theorem C.3. *If $\succeq$ has an expected utility representation, the following hold:*

(i) The decision maker is (strictly) risk averse if and only if the Bernoulli utility function is (strictly) concave.

(ii) The decision maker is (strictly) risk seeing if and only if the Bernoulli utility function is (strictly) convex.

(iii) The decision maker is risk neutral if and only if the Bernoulli utility function is linear.

Question C.10. *Prove and provide intuition for the following statement: even if $\succeq$ has an expected utility representation, the decision maker is not necessarily classified into one of the three types of risk attitudes. Provide an example.*

Question C.11. *Prove that the decision maker described in Question C.3 is risk neutral.*

Question C.12. *Prove and provide intuition for the following statements: the property that "the decision maker is risk averse" is ordinal. This is despite the fact that we characterize risk attitudes in terms of a cardinal representation (Theorem C.3).*

SECOND-ORDER STOCHASTIC DOMINANCE: Using our definition of risk aversion, we compare lotteries in terms of how risky they are, using the following criterion. We say that p second-order stochastically dominates q, if p is preferred to q by every risk averse decision maker, i.e., formally, this is the case if $\mathbb{E}_p(u) \geq \mathbb{E}_q(u)$ for every concave Bernoulli function $u : X \to \mathbb{R}$.

Question C.13. *Prove that lottery $p = (0.5 \times 30, 0.5 \times 10)$ second-order stochastically dominates lottery $q = (0.25 \times 40, 0.5 \times 20, 0.25 \times 0)$.*
> *Remark: q is called a mean-preserving spread of p, i.e., q is obtained by adding noise to each possible outcome that can be obtained under p. In this sense, q involves more uncertainty than p (which is why q is more risky than p).*

Question C.14. *Prove and provide intuition for the following statement: if p first-order stochastically dominates q, then it is also the case that p second-order stochastically dominates q. The converse is not necessarily true. Explain why and provide an example.*

RATIONALIZING CHOICE UNDER UNCERTAINTY: The revealed preference methodology can be used to test the vNM axioms. In particular, a choice correspondence C is rationalized by vNM preferences, if there exists a Bernoulli function $u : X \to \mathbb{E}$ such that, for each choice problem $A \in \mathcal{E}$,

$$C(A) = \{p \in A : \mathbb{E}_p(u) \geq \mathbb{E}_q(u) \text{ for all } q \in A\}.$$

And of course, in a similar way we can define rationalizing a choice function.

Question C.15. (ALLAIS PARADOX). *Consider the following lotteries $p_1 = (0.25 \times 3000, 0.75 \times 0)$, $p_2 = (0.2 \times 4000, 0.8 \times 0)$, $q_1 = (1 \times 3000)$ and $q_2 = (0.8 \times 4000, 0.2 \times 0)$. An experimental subject participates in the experiment $\mathcal{E} = \{\{p_1, p_2\}, \{q_1, q_2\}\}$ and chooses according to the choice correspondence $C(\{p_1, p_2\}) = \{p_2\}$ and $C(\{q_1, q_2\}) = \{q_1\}$. Can this data be rationalized by vNM preferences? What do you think most subject choose?*

Question C.16. *Consider the experiment $\mathcal{E} = \{\{p_1, p_2\}, \{q_1, q_2\}, \{r_1, r_2\}\}$, where $p_1 = (1 \times 10)$, $p_2 = (0.5 \times 20, 0.5 \times 0)$, $q_1 = (1 \times 20)$, $q_2 = (\frac{1}{3} \times 80, \frac{2}{3} \times 10)$, $r_1 = (1 \times 80)$ and $r_2 = (\frac{1}{3} \times 160, \frac{2}{3} \times 20)$. Suppose that the experimental subject chooses according to the choice correspondence $C(\{p_1, p_2\}) = \{p_1, p_2\}$, $C(\{q_1, q_2\}) = \{q_1, q_2\}$ and $C(\{r_1, r_2\}) = \{r_1, r_2\}$. Can this data be rationalized by vNM preferences? If yes, provide an appropriate Bernoulli utility function.*

C.2 Subjective uncertainty

SET OF STATES: (finitely many) contingencies that the individual is uncertain about. There are no objective probabilities describing this uncertainty. Each state is denoted by ω and the set of all states is denoted by $\Omega = \{\omega_1, \ldots, \omega_N\}$.

ACT: it is a contract $f : \Omega \to \mathcal{L}$ that pays one lottery at each state. Thus, $f(\omega)$ is the lottery that the individual will receive (according to the act f) if the state ω is realized. Acts can also be written as vectors of lotteries, i.e., $f = (p_1, \ldots, p_N)$ where $p_n = f(\omega_n)$ for each $n = 1, \ldots, N$. Overall, the idea is that the act is a two-stage procedure: first the state is realized and then an outcome from the corresponding lottery is drawn. The set of all acts is denoted by $\mathcal{F}$. Each lottery p can be seen as a constant act that pays $f(\omega) = p$ at every state $\omega \in \Omega$, i.e., in other words, from the decision maker's point of view the act $(p, \ldots, p) \in \mathcal{F}$ and the lottery $p \in \mathcal{L}$ are essentially the same thing. In this sense, with slight abuse of notation, we sometimes write $\mathcal{L} \subseteq \mathcal{F}$.

PREFERENCES OVER ACTS: complete and transitive preferences over $\mathcal{F}$. Strict preferences and indifference are defined as above. Since $\mathcal{L} \subseteq \mathcal{F}$, preferences over acts induce vNM preferences over lotteries, i.e., $p \succeq q$ if and only if $(p, \ldots, p) \succeq (q, \ldots, q)$. The induced preferences over $\mathcal{L}$ are assumed to be vNM, i.e., they have a vNM EU representation. Thus, the act $f \in \mathcal{F}$ yields expected utility $\mathbb{E}_{f(\omega)}(u)$ at each state $\omega \in \Omega$. Finally, induced preferences over X are assumed to be monotonic.

Question C.17 (Maxmin preferences). *There are two states identified by the party that wins the next american presidential elections, i.e., $\Omega = \{Democrat\ (\omega_1), Republican\ (\omega_2)\}$. A decision maker compares any two acts f and g on the basis of the worst case scenario, i.e.,*

$$f \succeq g \Leftrightarrow \min\{\mathbb{E}_{f(\omega_1)}(u), \mathbb{E}_{f(\omega_2)}(u)\} \geq \min\{\mathbb{E}_{g(\omega_1)}(u), \mathbb{E}_{g(\omega_2)}(u)\}.$$

Verify that this preference relation is complete and transitive.

Question C.18 (Only one state matters). *There are two states identified by the party that wins the next american presidential elections, i.e., $\Omega = \{Democrat\ (\omega_1), Republican\ (\omega_2)\}$. A decision maker compares any two acts f and g on the basis of the lottery they induce in the Democrat wins, i.e.,*

$$f \succeq g \Leftrightarrow f(\omega_1) \succeq g(\omega_1).$$

Verify that this preference relation is complete and transitive. Verify that the induced preferences over $\mathcal{L}$ are vNM.

UTILITY REPRESENTATION: the preferences over acts are represented by a function $u : \mathcal{F} \to \mathbb{R}$ such that, for every pair of acts $f, g \in \mathcal{F}$,

$$f \succeq g \Leftrightarrow u(f) \geq u(g).$$

SUBJECTIVE BELIEFS: a probability distribution μ over Ω, with $\mu(\omega)$ denoting the probability that the individual attaches to the state $\omega \in \Omega$. These probabilities are not objectively given (unlike for instance the probabilities involved in a lottery). Rather they can be seen as a representation of how likely the decision maker deems each state. Such beliefs are of course unobservable by the economist.

SUBJECTIVE EXPECTED UTILITY: Once again, we are particularly interested in utility functions that can be written as expectations. Formally, this is done by first assigning Bernoulli utilities to

the sure outcomes via the function $u : X \to \mathbb{R}$, then assigning beliefs to each state via probability distribution μ, and only then assign to each act $f = (p_1, \dots, p_N) \in \mathcal{L}$ (subjective) expected utility

$$\mathbb{E}_\mu(u(f)) = \sum_{\omega \in \Omega} \mu(\omega)\mathbb{E}_{f(\omega)}(u) = \sum_{n=1}^{N} \mu(\omega_n)\mathbb{E}_{p_n}(u).$$

The idea is that we first compute the expected utility of the lottery at each state ω, and then we take the expectation of these expected utilities with respect to the subjective beliefs. Note that once we have assigned Bernoulli utilities to the different outcomes in X and subjective probabilities to the different states in Ω, the subjective expected utility of each act follows automatically, and we do not have any flexibility in adjusting it. Then, the question becomes whether we can always find Bernoulli utilities and beliefs, in a way such that for every $f, g \in \mathcal{F}$,

$$f \succeq g \Leftrightarrow \mathbb{E}_\mu(u(f)) \geq \mathbb{E}_\mu(u(g)).$$

If this is the case, we say that the preferences $\succeq$ have an Anscombe-Aumann (AA) subjective expected utility representation.

Question C.19. *Do the preferences in Questions C.17 and C.18 have a subjective expected utility representation? If yes, identify appropriate Bernoulli utilities and subjective beliefs.*

AA AXIOMS: There are axioms that guarantee the existence of a SEU representation, viz., the vNM axioms that we have already discussed, plus two new axioms, called (A_5) AA monotonicity and (A_6) non-triviality (we will elaborate on AA monotonicity later on).

Theorem C.4 (AA SEU Theorem). *The preferences $\succeq$ over $\mathcal{F}$ have an AA SEU representation if and only if they satisfy completeness, transitivity, continuity, independence, AA monotonicity and non-triviality.*

DOMINANCE RELATIONS: There are two forms of dominance, both based on the idea of statewise comparing the lotteries of f with those of g.

- STRICT DOMINANCE: f strictly dominates g whenever $f(\omega) \succ g(\omega)$ for every $\omega \in \Omega$.

- WEAK DOMINANCE: f weakly dominates g whenever $f(\omega) \succeq g(\omega)$ for every $\omega \in \Omega$ with the preference being strict in at least one state.

Note that both notions of dominance depend on the induced preferences over $\mathcal{L}$, without making explicit reference to the beliefs. It is also not difficult to verify that both dominance relations are incomplete, i.e., not all pairs of acts can be ordered on the basis of one dominating the other.

Question C.20. *Prove the following statement: if f strictly dominates g, then $f \succ g$. Explain why this is true irrespective of the decision maker's beliefs. Show by means of an example that the converse is not true.*

Question C.21. *Prove the following statement: for any full-support belief (i.e., μ such that $\mu(\omega) > 0$ for all $\omega \in \Omega$) if f weakly dominates g, then $f \succ g$. Show by means of an example that the converse is not true.*

MIXED ACT: For any two acts $f, g \in \mathcal{F}$ and any $\alpha \in (0, 1)$, we define the mixed act $(\alpha \times f, (1-\alpha) \times g)$ yields at each state $\omega \in \Omega$ the lottery that draw each $x \in X$ with probability $\alpha f(\omega)(x) + (1 - \alpha)g(\omega)(x)$. The idea is that the decision maker randomizes at each state between the lottery induced by f and the lottery induced by g.

Question C.22 (Challenging). *Consider the following three acts that pay different amounts depending on whether a Democrat or a Republican candidate wins the next american presidential elections.*

act	Democrat	Republican
f	$10	$0
g	$0	$10
h	$5	$5

Consider an individual with preferences over acts that have a SEU representation. Note that we do not know the individual's beliefs over the set of states. Prove formally and provide intuition for the following statements:

(a) *If the individual is risk-seeking, then there exists some $\alpha \in (0, 1)$ such that the mixed act $(\alpha \times f, (1 - \alpha) \times g)$ strictly dominates h.*

(b) *If the individual is risk-averse, then there exists a belief μ such that h is a rational choice in the decision problem $A = \{f, g, h\}$.*

Problem C.2 (Very challenging). *A forecaster is asked to report his subjective belief of the Democratic candidate winning the next american presidential elections, and reports a probability $r \in [0, 1]$, which of course we do not know if it is the same as his true belief. The forecaster is compensated depending on his reported belief and the realized state (i.e., based on what he said and who won). In particular, if the Democrat is elected then he wins $10,000 with probability $1 - (1 - r)^2$, whereas if the Republican is elected then wins $10,000 with probability $1 - r^2$.*

(a) *Write each report $r \in [0, 1]$ as an act.*

(b) *If his preferences over acts have a SEU representation, prove that his unique optimal choice is to report his true belief. Which is the implication of this result?*

RATIONALIZING CHOICE UNDER UNCERTAINTY: The revealed preference methodology can be used to test the AA axioms. In particular, a choice correspondence C is rationalized by AA preferences, if there exists a Bernoulli function $u : X \to \mathbb{E}$ and a belief μ such that, for each choice problem $A \in \mathcal{E}$,

$$C(A) = \{f \in A : \mathbb{E}_\mu(u(f)) \geq \mathbb{E}_\mu(u(g)) \text{ for all } g \in A\}.$$

And of course, in a similar way we can define rationalizing a choice function.

Question C.23. (ELLSBERG PARADOX). *Consider an urn containing three balls. An experimental subject is told that exactly one ball is red and the remaining two balls are either (i) both black, or (ii) one is black and the other one is yellow, or (iii) both yellow. A ball will be randomly drawn from the earn. Consider the following alternatives:*

f_1 : *if the ball is red you get 10 Euros*

f_2 : *if the ball is black you get 10 Euros*

g_1 : *if the ball is red or yellow you get 10 Euros*

g_2 : *if the ball is black or yellow you get 10 Euros*

First, define the states and write the acts that correspond to the four alternative aboves. Then, take the experiment $\mathcal{E} = \{\{f_1, f_2\}, \{g_1.g_2\}\}$, and consider a subject who chooses according to the choice correspondence $C(\{f_1, f_2\}) = \{f_1\}$ and $C(\{g_1, g_2\}) = \{g_2\}$. Can this dataset be rationalized by AA preferences? What do you think most subject choose?

Part II
Game Theory

D   Strategic-form games

GAMES: This is a term that we use for decision problems with interacting decision makers, i.e., there are multiple decision makers and the outcome depends on the choices of all of them. This type of interaction is called strategic interaction. Throughout the course, unless explicitly stated otherwise we will consider games with only two decision makers. We classify games in two categories, strategic-form games (where everybody acts simultaneously) and extensive-form games (where decision makers act sequentially). The essential difference between the two pertains in the information that decision makers have when the choose actions.

STRATEGIC-FORM GAME: It is modelled by a tuple $\langle I, (A_i)_{i \in I}, (o_i)_{i \in I} \rangle$, where $I = \{\text{Ann } (a), \text{ Bob } (b)\}$ is the finite set of players, A_i is player i's finite set of actions, and $o_i : A_a \times A_b \to X$ induces an outcome for each player i and for each action profile (a_a, a_b). For simplicity, we will continue assuming that outcomes are monetary payoffs, i.e., $X \subseteq \mathbb{R}$. Hence, each action profile (a_a, a_b) yields a pair $\big(o_a(a_a, a_b), o_b(a_a, a_b)\big) \in \mathbb{R}^2$ of monetary payoffs.

Example D.1. Consider the following strategic form game, where $A_a = \{T, M, B\}$ and $A_b = \{L, R\}$.

	L	R
T	\$10 , \$5	\$0 , \$0
M	\$0 , \$5	\$10 , \$0
B	\$5 , \$0	\$5 , \$5

Each action profile yields a pair of monetary payoffs. The first number refers to Ann's payoff and the second one to Bob's. For instance, $\big(o_a(M, L), o_b(M, L)\big) = (\$0, \$10)$. ◁

GAMES AS DECISION PROBLEMS UNDER SUBJECTIVE UNCERTAINTY: From player i's point of view, each of the opponent's actions $a_j \in A_j$ can be seen as a state, and each own action $a_i \in A_i$ can be seen as an act that yields a pair of payoffs at each state. In this sense, a game can be seen as a decision problem under subjective uncertainty. This type of uncertainty is called strategic, because what i believes about what j is going play (i.e., i's beliefs about the state space A_j) typically depends on how i thinks that j reasons about what i is going to do.

PREFERENCES: Each player $i \in I$ has AA preferences $\succeq_i$ over the set of acts. Recall that a SEU representation of i's preferences over acts consists of a Bernoulli utility function over pairs of monetary payoffs ($u_i : X \times X \to \mathbb{R}$) and a belief over j's actions (μ_i). Like in decision theory under uncertainty, the preferences over acts induce preferences over pairs of payoffs $X \times X$. To simplify notation, we will henceforth write

$$(a_i, a_j) \succeq_i (a_i', a_j') \Leftrightarrow \big(o_i(a_i, a_j), o_j(a_i, a_j)\big) \succeq_i \big(o_i(a_i', a_j'), o_j(a_i', a_j')\big),$$

i.e., player i prefers the action profile (a_i, a_j) over (a_i', a_j') if and only if she prefers the monetary payoffs associated with (a_i, a_j) over those associated with (a_i', a_j').

Question D.1. *In the game of Example D.1, assume that both players are selfish. Then, provide three Bernoulli utility representations, one where they are both risk-neutral, one where they are both risk-averse and one where they are both risk-seeking. Do the three profiles of Bernoulli utility functions induce the same preferences over action profiles?*

SUBJECTIVE EXPECTED UTILITY: it is defined analogously to our models of decision theory under subjective uncertainty. Namely, given a Bernoulli utility function u_i and a belief μ_i, player i's SEU from an action a_i is equal to

$$\mathbb{E}_{\mu_i}(u_i(a_i)) := \sum_{a_j \in A_j} \mu_i(a_j) u_i(a_i, a_j).$$

This utility representation suggest that

$$a_i \succeq_i a_i' \Leftrightarrow \mathbb{E}_{\mu_i}(u_i(a_i)) \geq \mathbb{E}_{\mu_i}(u_i(a_i,)),$$

i.e., i prefers a_i to a_i' if and only if the SEU of the former is greater than the one of the latter. Clearly, which action is preferred over which depends in general on both the Bernoulli utility functions and on the beliefs about the opponent's actions.

RATIONALITY: given a belief μ_i, the action a_i of player i is a rational choice if, for all $a_i' \in A_i$,

$$\mathbb{E}_{\mu_i}(u_i(a_i)) \geq \mathbb{E}_{\mu_i}(u_i(a_i')).$$

That is, a_i is the most preferred actions among all those available in A_i. Again, notice that rationality is a relative concept in the sense that it depends on both the Bernoulli utilities and on the beliefs of player i. In this sense, figuring out which actions can be rationally chosen depends on the information that we have about the AA preferences. If we knew both the Bernoulli utility functions and the beliefs, we could pin down exactly the rational actions. If we only know only the Bernoulli utility functions but not the beliefs, we get more rational actions. And of course, if we know only the induced ordinal utilities, we end up with even more rational actions.

Question D.2. *Using each of the three Bernoulli utility functions from Question D.1 separately, find the actions of Ann that can be rationally be played for some belief in the game of Example D.1.*

METHODOLOGICAL APPROACH: it is not necessarily the case that others (i.e., the opponent or even the game-theorist who studies the game) can observe the Bernoulli utility functions or the beliefs of the players. So, we will distinguish two cases:

- ORDINAL GAMES: Only the induced preferences over allocations of monetary payoffs are commonly known.

- CARDINAL GAMES: The Bernoulli utility functions are also commonly known.

In neither of the two cases, do others know anything about the player's beliefs.

SOLUTION CONCEPT: it is a blackbox that takes as input the structure of the game (i.e., $\langle I, (A_i)_{i \in I}, (u_i)_{i \in I} \rangle$) and returns a collection of action profiles (i.e., a subset of $A_a \times A_b$) as prediction of what could be played. Importantly the prediction does not need to be unique. Solution concepts that we study in this course are classified as follows:

	ELIMINATION	EQUILIBRIUM
ORDINAL GAMES	Iterated Strict/Weak/Börgers Dominance	Nash Equilibrium
CARDINAL GAMES	Iterated Strict Dominance	Mixed Nash equilibrium

The row classification distinguishes games on the basis of the information that is available on the players' preferences, i.e., based on whether u_i is an ordinal or a Bernoulli utility function. So, the difference here is simply on the input side of the solution concept. The column classification distinguishes solution concepts on the basis of the players' strategic reasoning. So, the difference here lies in what happens inside the blackbox. All solution concepts assume that players are rational, and differ in the additional assumptions that they make.

D.1 Ordinal games

ORDINAL UTILITY FUNCTION: The only information that we have is the players' preferences over allocations of payoff pairs, or equivalently over $A_a \times A_b$. These preferences are represented by an ordinal utility function $u_i : A_a \times A_b \to \mathbb{R}$.

ELIMINATION SOLUTION CONCEPTS: Our approach will be to start eliminating actions that are not likely to be rational, given our (limited) information about the players' preferences. There are different ways to do this, corresponding to different assumptions on the players' beliefs, and eventually leading to different solution concepts.

STRICT DOMINANCE: The action $a_i \in A_i$ strictly dominates the action $a_i' \in A_i$ in the game $A_i \times A_j$, whenever for all $a_j \in A_j$,

$$u_i(a_i, a_j) > u_i(a_i', a_j).$$

The idea a_i is strictly preferred to a_i' irrespective of what the opponent does. Note that dominance is a notion which is relative to the game $A_i \times A_j$. Whenever it is obvious which game we are referring to, we will omit explicit reference. Otherwise, we will have to explicitly mention it. If there is an action a_i that strictly dominates a_i', we say that a_i' is strictly dominated. If there is an action a_i that strictly dominates every other a_i', we say that a_i is strictly dominant. Obviously, if a_i' is strictly dominated (by some a_i), it cannot be rationally chosen irrespective of the Bernoulli utility function and the beliefs that player i has (which are both unobservable). In this sense, we can be sufficiently certain that a_i' will not be chosen.

Question D.3. *Explain the following statement: A strictly dominated action is not rational (i.e., rationality is a sufficient condition for ruling out all strictly dominated actions).*

Question D.4. *Prove and provide intuition for the following statement: A strictly dominant action is the only rational action.*

Question D.5. (PUBLIC GOOD GAME). *Ann and Bob are initially endowed with 10 Euros each. Each of them independently chooses an amount from the set $\{0, 1, \ldots, 10\}$ to put into a common account. The remaining of their endowment will go into a private account. The public account has an interest rate of 50%, i.e., the total amount that goes into the public account is multiplies by 1.5. On the other hand, the private accounts do not have any interest. Each individual's total payoff will be equal to the amount in their private account plus half the amount in the public account. Assuming that both are selfish and rational, what do we expect them to play? Explain formally and intuitively.*

ITERATED STRICT DOMINANCE (ISD): For each step $k \geq 0$ and each player $i \in I$:

$$
\begin{aligned}
S_i^0 &:= A_i \\
S_i^1 &:= \{a_i \in S_i^0 : a_i \text{ is not strictly dominated in } S_i^0 \times S_j^0\} \\
S_i^2 &:= \{a_i \in S_i^1 : a_i \text{ is not strictly dominated in } S_i^1 \times S_j^1\} \\
&\vdots \\
S_i^k &:= \{a_i \in S_i^{k-1} : a_i \text{ is not strictly dominated in } S_i^{k-1} \times S_j^{k-1}\} \\
&\vdots
\end{aligned}
$$

Then, the actions of player i that survive ISD are those in $S_i := \bigcap_{k=0}^{\infty} S_i^k$. The predictions of ISD are the action profiles in $ISD := S_a \times S_b$.

WEAK DOMINANCE: The action $a_i \in A_i$ weakly dominates the action $a_i' \in A_i$ in the game $A_i \times A_j$, whenever

$$u_i(a_i, a_j) \geq u_i(a_i', a_j)$$

for all $a_j \in A_j$, with at least one of the inequalities being strict. If there is an action a_i that weakly dominates a_i', we say that a_i' is weakly dominated. If there is an action a_i that weakly dominates every other a_i', we say that a_i is weakly dominant. If a_i' is weakly dominated (by some a_i), it cannot be rationally chosen irrespective of the Bernoulli utility function, as long as the beliefs that player i are full-support. However, the converse is not necessarily true (Question C.21). In this sense, we can be sufficiently certain that a_i' will not be chosen, as long as we are confident that the players do not discard ex ante any of the opponents' strategies.

Question D.6. *Explain the following statement: A weakly dominated action may be rational (i.e., rationality is not a sufficient condition for ruling out weakly dominated actions, unless of course we assume that beliefs are full support).*

Question D.7. (SECOND-PRICE AUCTION). *Ann and Bob participate in an auction for painting. Each of them has a budget of $100k. Ann's private value for the object is $40k and Bob's private value is $30k. The rules of the auction are as follows: (a) each participant can bid multiples of $1k, (b) the winner is the participant with the highest bid, and in case of a tie Ann is the winner, (c) the winner pays a price equal to the bid of the other participant (i.e., the second price). Explain the following statement: it is a weakly dominant action for Ann to bid $40k. Then, provide an example of some action (other than bidding her own private value) which can be rationally played by Ann.*

Question D.8. *Prove and provide intuition for the following statement: A weakly dominant action is always rational (irrespective of the Bernoulli utility function and the beliefs).*

ITERATED WEAK DOMINANCE (IWD): For each step $k \geq 0$ and each player $i \in I$:

$$
\begin{aligned}
W_i^0 &:= A_i \\
W_i^1 &:= \{a_i \in W_i^0 : a_i \text{ is not weakly dominated in } W_i^0 \times W_j^0\} \\
W_i^2 &:= \{a_i \in W_i^1 : a_i \text{ is not weakly dominated in } W_i^1 \times W_j^1\} \\
&\vdots \\
W_i^k &:= \{a_i \in W_i^{k-1} : a_i \text{ is not weakly dominated in } W_i^{k-1} \times W_j^{k-1}\} \\
&\vdots
\end{aligned}
$$

Then, the actions of player i that survive IWD are those in $W_i := \bigcap_{k=0}^{\infty} W_i^k$. The predictions of IWD are the action profiles in $IWD := W_a \times W_b$.

Question D.9. *Explain the following statement: If a player has a strictly dominant strategy, we do not need to impose any assumption about her strategic reasoning. On the other hand, solution concepts that are based on iteratively eliminating dominated strategies, make implicit assumptions about the players' strategic reasoning.*

Question D.10. (GUESSING GAME). *This is one of the very few instances where we will have games with more than two players. There are n experimental subjects, and each of them is asked to chooses an number from the set $\{1, 2, \ldots, 100\}$. We then compute the average, and whoever is closer to 2/3 of the average wins a prize of $100. If there are several winners, they split the prize. If everybody is selfish, what does IWD predict? What do you think most subject do? What do we conclude then?*

BÖRGERS DOMINANCE: As we have discussed, strict dominance is too permissive, i.e., it eliminates some actions that are not rational (Question D.3), but there may exist additional actions that cannot be rationally chosen. On the other hand, weak dominance is too restrictive, i.e., it may eliminate actions that are rational (Question D.6). Thus, we naturally ask, is there an elimination concept (in between the two) that eliminates exactly those actions that cannot be rationally chosen for any Bernoulli utility function and any belief. In other words, we are looking for an elimination procedure that constitutes a necessary and sufficient condition for rationality. This is exactly what Börgers dominance does: The action $a_i \in A_i$ is Börgers dominated in the game $A_i \times A_j$, whenever for every nonempty $A_j' \subseteq A_j$ there exists some $a_i' \in A_i$ (which may depend on A_j') such that a_i' weakly dominates a_i in $A_i \times A_j'$.

Theorem D.1. *For an action $a_i \in A_i$ the following statements are equivalent:*

(a) *a_i is Börgers dominated in $A_i \times A_j$.*

(b) *There is no Bernoulli utility function (inducing the ordinal preferences over $A_i \times A_j$) and belief μ_i such that a_i is rational given μ_i.*

Problem D.1. *In the following game, assume that both players are selfish.*

	L	R
T	$10 , $5	$0 , $0
M	$0 , $5	$1 , $0
B	$9 , $0	$0 , $5

Show that action B is Börgers dominated, while M is not Börgers dominated. Then, prove that there is no Bernoulli utility function u_a and no belief μ_a such that B is rational in $\{T, M, B\} \times \{L, R\}$, but there exists a Bernoulli utility function u_a and a belief μ_a such that M is rational in $\{T, M, B\} \times \{L, R\}$.

ITERATED BÖRGERS DOMINANCE (IBD): For each step $k \geq 0$ and each player $i \in I$:

$$
\begin{aligned}
B_i^0 &:= A_i \\
B_i^1 &:= \{a_i \in B_i^0 : a_i \text{ is not Börgers dominated in } B_i^0 \times B_j^0\} \\
B_i^2 &:= \{a_i \in B_i^1 : a_i \text{ is not Börgers dominated in } B_i^1 \times B_j^1\} \\
&\ \ \vdots \\
B_i^k &:= \{a_i \in B_i^{k-1} : a_i \text{ is not Börgers dominated in } B_i^{k-1} \times B_j^{k-1}\} \\
&\ \ \vdots
\end{aligned}
$$

Then, the actions of player i that survive IBD are those in $B_i := \bigcap_{k=0}^{\infty} B_i^k$. The predictions of IBD are the action profiles in $IBD := B_a \times B_b$.

Question D.11. *Prove formally that $IWD \subseteq IBD \subseteq ISD$. Explain what it says.*

EQUILIBRIUM SOLUTION CONCEPTS: The nice thing with elimination solution concepts is that each player can run the algorithm in their head and come up with the solution without needing any prior information about the opponent's play. This makes them reasonable (from a logical point of view) but oftentimes quite demanding (in terms of the reasoning that they require players to undertake). On the other hand equilibrium concepts, implicitly assume that each player can correctly predict the opponent's action, and respond rationally to it. This can be often justified as the steady state of some learning process.

BEST RESPONSE: This is the main notion used to define an equilibrium concept. The action $a_i \in A_i$ is a best response to $a_j \in A_j$, if for all $a_i' \in A_i$,

$$
u_i(a_i, a_j) \geq u_i(a_i', a_j).
$$

Importantly, we do not compare a_i with a_i' for every action of the opponent, but only for a_j. There may exist multiple best responses to a_j. The set of all best responses to a_j will be denoted by $BR_i(a_j)$. So, if a_i is a best response to a_j we write $a_i \in BR_i(a_j)$.

Question D.12. *Prove and provide intuition for the following statements:*

(a) If a_i is a best response to some a_j then it is not Börgers dominated (and a fortiori not strictly dominated), but it may still be weakly dominated.

(b) If a_i is strictly dominant then it is the only best response to each $a_j \in A_j$.

(c) If a_i is weakly dominant then it is the one of the (perhaps multiple) best responses to each $a_j \in A_j$.

NASH EQUILIBRIUM: the action profile (a_a, a_b) is a Nash equilibrium if $a_i \in BR_i(a_j)$ for both $i \in I$. That is, none of the players has an incentive to unilaterally deviate from this action profile. Another interpretation is that both players are choose a rational action and at the same time they can correctly guess the opponent's action. The set of all Nash Equilibria is denoted by $NE \subseteq A_a \times A_b$.

Question D.13. *There are two firms that serve a market, one owned by Ann and one by Bob. The two firms simultaneously choose a price each (p_a and p_b respectively) from the set $\{1, 2, 3, 4\}$. If one of the two firms charges a lower price than the opponent, this firm will serve the entire market, i.e.,*

the market price will be $p = \min\{p_a, p_b\}$ and the firm with the lowest price will sell $Q(p) := 16 - 4p$ units of the product whereas the other firm will sell 0 units. If the two firms charge the same price, the market price is $p = p_a = p_b$ and they share the market, i.e., each of the two firms sells $Q(p)/2$ units. Suppose that the cost per unit for Ann is equal to $c_a = 1$ and for Bob is equal to $c_b = 2$. Both firms are selfish in the sense that they want to maximize their own profits. Find the Nash Equilibria of this game.

Question D.14. *Take the following three very well-known games, assuming that the utilities are Bernoulli. Nevertheless, only ordinal the preferences over action profiles are commonly known.*

	D	C
D	2 , 2	7 , 0
C	0 , 7	5 , 5

Prisoner's Dilemma

	O	B
O	3 , 1	0 , 0
B	0 , 0	1 , 3

Battle of the Sexes

	H	T
H	1 , -1	-1 , 1
T	-1 , 1	1 , -1

Matching Pennies

Find the Nash Equilibria in each of the corresponding ordinal games.

Problem D.2. (FIRST-PRICE AUCTION). *Ann and Bob participate in an auction for a painting. Each of them has a budget of \$100k. Ann's private value for the object is \$40k and Bob's private value is \$30k. The rules of the auction are as follows: (a) each participant can bid multiples of \$1k, (b) the winner is the participant with the highest bid, and in case of a tie Ann is the winner, (c) the winner pays a price equal to his/her own bid (i.e., the first price). Is it a Nash Equilibrium if both players bid their own private value? Find a Nash Equilibrium where one of the players bids above his/her private value.*

Question D.15. *Prove and provide intuition for the following statement: Every Nash Equilibrium survives IBD (and a fortiori it also survives ISD). The converse is not necessarily true, i.e., not all action profiles surviving IBD are always NE (and a fortiori not all action profiles surviving ISD are always NE).*

Question D.16. *The previous statement does not extend to IWD, i.e., a Nash Equilibrium may not survive IWD. Provide an example where a Nash Equilibrium is eliminated by IWD.*

D.2 Cardinal games

BERNOULLI UTILITY FUNCTION: We now have information about the players' vNM preferences for lotteries over allocations of payoffs, i.e., we know the Bernoulli utility function $u_i : A_a \times A_b \to \mathbb{R}$ of each player $i \in I$.

MIXED ACTION: The fact that we can now attach a Bernoulli utility to each payoff allocation means that we can introduce probabilities and take expectations. A mixed action σ_i is a probability distribution over the player's actions, which attaches probability $\sigma_i(a_i)$ to each $a_i \in A_i$. The probability distribution σ_i is chosen by player i herself. Intuitively, the player delegates her action to a randomizing device, whose probabilities are objectively known. Obviously, an action $a_i \in A_i$ can be seen

as a (degenerate) mixed strategy σ_i such that $\sigma_i(a_i) = 1$. The set of all mixed actions of player i is denoted by $\mathcal{L}(A_i)$.

EXPECTED UTILITY FROM A MIXED ACTION: Following the interpretation that from i's point of view the game can be seen as a decision problem under subjective uncertainty, a mixed action is then seen as a contract that pays a lottery (of payoff allocations) at each state (i.e, for each action of the opponent). Thus, a mixed action induces an vNM expected utility for each action of the opponent. Formally, for a mixed action $\sigma_i \in \mathcal{L}(A_i)$ and each action $a_j \in A_j$, player i's expected utility is

$$u_i(\sigma_i, a_j) := \sum_{a_i \in A_i} \sigma_i(a_i) u_i(a_i, a_j).$$

Recall that player i has AA preferences over acts, which are represented by the Bernoulli utility function (which we can observe) and the beliefs μ_i over A_j (which we cannot observe). Hence, from i's point of view, the subjective expected utility from a mixed action σ_i is equal to

$$\mathbb{E}_{\mu_i}(u_i(\sigma_i)) := \sum_{a_j \in A_j} \mu_i(a_j) u_i(\sigma_i, a_j).$$

Of course, as we have mentioned, each action a_i can be seen as a degenerate mixed action that puts probability 1 to a_i, and therefore the subjective expected utility from the action a_i is given by simply replacing σ_i with a_i in the previous formula (see formula for $\mathbb{E}_{\mu_i}(u_i(a_i))$ in the beginning of this section).

Question D.17. *Using each of the three vNM utility functions from Question D.1 separately, find the Ann's expected utilities from a mixed action σ_a such that puts probability 1/2 to T and probability 1/4 to M and 1/4 to B.*

RATIONAL CHOICE: The introduction of mixed actions essentially enlarges the decision problem of player by introducing additional acts, i.e., formally, instead of presenting player i with the choice problem A_i, we are now presenting her with the choice problem $\mathcal{L}(A_i)$. Obviously, a choice σ_i is rational in this enlarged choice problem if it maximizes the player's subjective expected utility given her beliefs, i.e., if for all $\sigma_i' \in \mathcal{L}(A_i)$,

$$\mathbb{E}_{\mu_i}(u_i(\sigma_i)) \geq \mathbb{E}_{\mu_i}(u_i(\sigma_i')).$$

Then of course, if the rational choice σ_i is picked, an action from $\text{supp}(\sigma_i)$ will be observed, i.e., an action that receives positive probability by σ_i. Thus, we naturally ask: are there any serious consequences for the rational choices that we may observe, if we enlarge the game by allowing for mixed actions? As it turns out, not really!

Theorem D.2. *For any belief μ_i and mixed action σ_i, the following statements are equivalent:*

(a) *σ_i is rational in $\mathcal{L}(A_i)$, i.e., $\mathbb{E}_{\mu_i}(u_i(\sigma_i)) \geq \mathbb{E}_{\mu_i}(u_i(\sigma_i'))$ for all $\sigma_i' \in \mathcal{L}(A_i)$.*

(b) *every $a_i \in \text{supp}(\sigma_i)$ is rational in A_i, i.e., $\mathbb{E}_{\mu_i}(u_i(a_i)) \geq \mathbb{E}_{\mu_i}(u_i(a_i'))$ for all $a_i' \in A_i$.*

What the previous theorem says is that we can first simply look for the rational choices in the original problem A_i. Then, the mixed actions σ_i's that distribute probability only to these rational actions that you have just identified are the rational choices in the enlarged choice problem $\mathcal{L}(A_i)$. In this sense, whether you allow for mixed actions or not does not really matter, as in the end the rational

choices that you might observe are exactly the same. So, why do we need to introduce mixed actions? There are two reasons. First, the game theorist does not really know whether the player has access to a randomizing device or not. But fortunately, the previous theorem tells us that this lack of knowledge is not a major problem. Second, as we are going to see in a bit, mixed actions provide us with a very useful computational tool that allows us to identify the rational actions even when we do not now the player's beliefs.

STRICT DOMINANCE: A mixed action $\sigma_i \in \mathcal{L}(A_i)$ strictly dominates the action $a_i \in A_i$ in the game $A_i \times A_j$ whenever for all $a_j \in A_j$,
$$u_i(\sigma_i, a_j) > u_i(a_i, a_j).$$

The idea is exactly the same as the one we introduced in the section on decision theory under subjective uncertainty, i.e., σ_i induces at every state a lottery that is strictly preferred to the lottery that is induced by a_i. In other words, all statewise comparisons of σ_i and a_i yield a result strictly in favor of σ_i. If there exists a mixed action σ_i that strictly dominates a_i then we say that a_i is strictly dominated.

Question D.18 (Challenging). *Using each of the three Bernoulli utility functions from Question D.1 separately, find the actions of Ann that are strictly dominated in the game of Example D.1.*

The following important result identifies exactly the actions that can be rationally chosen from A_i by a rational player with (the known) Bernoulli utility function u_i and (some unknown) beliefs μ_i.

Theorem D.3. *For an action $a_i \in A_i$ and a known Bernoulli utility function u_i, the following two statements are equivalent:*

(a) *There exists some mixed action σ_i that strictly dominates a_i (given u_i).*

(b) *There is no belief μ_i such that a_i is rational given μ_i (and u_i).*

This clarifies the second reason why mixed actions are useful, i.e., they can help the game theorist to find the actions that can be expected to be played when the beliefs of the player are unknown.

Question D.19. *Rewrite the previous theorem in the following form: the negation of (a) is equivalent to the negation of (b).*

Question D.20. *Using Theorem D.3, explain that Question D.2 is equivalent to Question D.18.*

Question D.21 (Challenging, but conceptually very important!). *Explain the following statement: If an action a_i is strictly dominated in an ordinal game, then it is also strictly dominated in every cardinal game with the same induced preferences over $A_i \times A_j$. The converse is not necessarily true. This means that when we eliminate a strictly dominated action in the ordinal game, we are sure that this action is not rational in the cardinal game, but at the same time we may keep too many actions (i.e., we may keep actions that are not rational for any belief in the cardinal game). In particular, the following logical relations hold:*

$$a_i \text{ is strictly dominated in the cardinal game } \Leftrightarrow a_i \text{ is not rational} \qquad \text{(Theorem D.3)}$$
$$a_i \text{ is strictly dominated in the ordinal game } \Rightarrow a_i \text{ is not rational} \qquad \text{(Question D.3)}$$

That is, strict dominance in the cardinal game provides us with more refined information on what we should expect to see from a SEU maximizer, which is of course natural to expect given that in the cardinal game we know more about the player (i.e., we know the Bernoulli utilities).

Question D.22 (Challenging, but conceptually very important!). *Explain the following statement: When we eliminate a Börgers dominated action in the ordinal game, we are sure that this action is not rational in the cardinal game, but at the same time we may keep too many actions (i.e., we may keep actions that are rational given some other Bernoulli utility function). In particular, the following logical relations hold:*

a_i *is strictly dominated in the cardinal game* $\Leftrightarrow$ a_i *is not rational (known u_i)* (Theorem D.3)

a_i *is Börgers dominated in the ordinal game* $\Leftrightarrow$ a_i *is not rational (unknown u_i)* (Theorem D.1)

Once again, this is because the cardinal game contains more information about the player's preferences (i.e., we know the Bernoulli utilities).

Problem D.3. *Take the game from Example D.1 and assume that Ann is selfish. For starters, prove that B is not Börgers dominated (and a fortiori it is not strictly dominated in the ordinal game). Then, for each of the Bernoulli utility functions from Question D.1, find Ann's actions that can be rationally played. Relate your answers to Questions D.21 and D.22.*

ITERATED STRICT DOMINANCE (ISD): The definition is identical to the one for ordinal games, except for the fact that the notion of strict dominance is now adjusted to allow for mixed strategies. For each step $k \geq 0$ and each player $i \in I$:

$$
\begin{aligned}
S_i^0 &:= A_i \\
S_i^1 &:= \{a_i \in S_i^0 : a_i \text{ is not strictly dominated in } S_i^0 \times S_j^0\} \\
S_i^2 &:= \{a_i \in S_i^1 : a_i \text{ is not strictly dominated in } S_i^1 \times S_j^1\} \\
&\vdots \\
S_i^k &:= \{a_i \in S_i^{k-1} : a_i \text{ is not strictly dominated in } S_i^{k-1} \times S_j^{k-1}\} \\
&\vdots
\end{aligned}
$$

Then, the actions of player i that survive ISD are those in $S_i := \bigcap_{k=0}^{\infty} S_i^k$. The predictions of ISD are the action profiles in $ISD := S_a \times S_b$.

Question D.23 (Traveller's dilemma). *An airline loses two identical suitcases belonging to two different risk-neutral travellers. The airline manager explains that the airline is liable for a maximum of \$6 per suitcase, and in order to determine the actual value of the suitcase he asks them to write down the amount of their value at no less than \$2 and no larger than \$6 (only whole amounts are allowed). If both write down the same number, he will treat that number as the true value of both suitcases and reimburse both travellers that amount. If one writes down a smaller number than the other, this smaller number will be taken as the true value, and both travellers will receive that amount along with a bonus/malus: \$2 extra will be paid to the traveler who wrote down the lower value and a \$2 deduction will be taken from the person who wrote down the higher amount. Find the action profiles that survive ISD in this game.*

BEST RESPONSE: The notion of equilibrium is entirely analogous to the one in ordinal games. The only difference is that now we allow for mixed actions. A mixed action σ_i is a best response to σ_j, if for all $\sigma_i' \in \mathcal{L}(A_i)$

$$\mathbb{E}_{\sigma_j}(u_i(\sigma_i)) \geq \mathbb{E}_{\sigma_j}(u_i(\sigma_i')).$$

The set of all best responses to σ_j will be denoted by $BR_i(\sigma_j)$.

Question D.24. *Using Theorem D.2, prove that $\sigma_i \in BR_i(\sigma_j)$ if and only if*

$$\mathbb{E}_{\sigma_j}(u_i(a_i)) \geq \mathbb{E}_{\sigma_j}(u_i(a_i'))$$

for every $a_i \in \text{supp}(\sigma_i)$ and every $a_i' \in A_i$.

NASH EQUILIBRIUM: The action profile (σ_a, σ_b) is a (mixed) Nash equilibrium if $\sigma_i \in BR_i(\sigma_j)$ for both $i \in I$. A pure Nash Equilibrium is a degenerate $(\sigma_a, \sigma_b) \in NE$ such that $\sigma_a(a_a) = \sigma_b(a_b) = 1$ for a profile of actions (a_a, a_b). The set of all Nash Equilibria is denoted by $NE \subseteq \mathcal{L}(A_a) \times \mathcal{L}(A_b)$. The following result provides a third (historical) reason for using mixed actions.

Theorem D.4. *At least one mixed Nash Equilibrium always exists.*

Question D.25. *Explain the following statement: if (σ_i, σ_j) is a mixed Nash Equilibrium, then every $a_i \in A_i$ such that $\sigma_i(a_i) > 0$ survives ISD.*

Question D.26. *Find the Mixed Nash Equilibria of "Battle of the Sexes" and "Matching Pennies" from Question D.14.*

Problem D.4. *Find the Mixed Nash Equilibria in the following game:*

	L	M	R
T	7 , 1	0 , 1	6 , 0
M	0 , 1	10 , 1	4 , 0
B	4 , 0	3 , 0	5 , 5

Question D.27. *Mixed Nash Equilibrium is often tested using aggregate data. This means that we observe n independent plays of the same game, and we check if the empirical frequency of play for the role of player a versus the empirical frequency of play for the role of player b form a Mixed Nash Equilibrium. What is the implicit underlying assumption that we make here if all observations come from the same pair of subjects playing against each other? Which is the additional implicit assumption that we make when the different observations come from different pairs of subjects?*

Question D.28. *Explain the following statements intuitively: in a cardinal game, if we take a strictly increasing linear transformation of the Bernoulli utilities, both ISD and NE remain unchanged. On the other hand in an ordinal game, if we take any strictly increasing transformation (not necessarily linear) of the ordinal utility functions, ISD, IWD, IBD and NE remain all unchanged.*

E   Extensive-form games

GAME TREE: a finite (directed) graph $(\bar{H}, \mathcal{E})$, where $\bar{H}$ is the set of all nodes and $\mathcal{E} := \{(h, h') \in \bar{H} \times \bar{H} : h'$ is a direct successor of $h\}$. We say that h'' is a successor of h if there is a directed path from h to h'', i.e., if there exists a sequence of nodes $(h_1, \ldots, h_N)$ such that (a) $h_1 = h$, (b) $h_N = h''$, and (c) $(h_n, h_{n+1}) \in \mathcal{E}$ for all $n = 1, \ldots, N - 1$. The set of weak successors of h contains all its successors plus h itself. Whenever h' is a direct successor of h, we say that h is a direct predecessor of h'. The root $h_0 \in \bar{H}$ is the only node that does not have any direct predecessor. Every other node has exactly one direct predecessor. Those nodes that do not have any direct successor are called terminal nodes and are denoted by $Z \subseteq \bar{H}$. All non-terminal nodes are denoted by $H := \bar{H} \setminus Z$.

ACTIVE PLAYERS: The set of players is still assumed to be $I = \{$Ann (a), Bob $(b)\}$. There is a function $p : H \to I$, assigning a player $p(h)$ to every non-terminal node. This is the player who decides to which direct successor of h we will move if h is reached. The set of nodes where player i is active is denoted by H_i.

ACTIONS: At a non-terminal node $h \in H_i$, the active player's set of actions are identified by the direct successors, and are denoted by $A_i(h)$. That is, each $a_i \in A_i(h)$ leads to some $h' \in \bar{H}$ such that $(h, h') \in \mathcal{E}$. For simplicity, we assume that $A_i(h)$ has at least two actions.

OUTCOMES: There is an outcome function $o_i : Z \to X$ for each player $i \in I$. For simplicity, we keep assuming that $X \subseteq \mathbb{R}$.

PREFERENCES AND UTILITIES: Each player $i \in I$ is assumed to have preferences $\succeq_i$ over allocations of playoffs $(o_a(z), o_b(z)) \in \mathbb{R}^2$. Similarly to strategic-form games. For the most part, we will consider ordinal preferences over allocations of payoffs. Hence, for each player $i \in I$ there is an ordinal utility function $u_i : Z \to \mathbb{R}$ representing $\succeq_i$. We will say that a game is without ties if, for every player $i \in I$ and every two terminal nodes $z, z' \in Z$ it is the case that $u_i(z) \neq u_i(z')$. Very importantly, these preferences remain stable throughout the game.

Example E.1. Ann starts by proposing a way to split of \$2. She can offer \$0 or \$1 or \$2 to Bob. If she offers \$1 or \$2, Bob can either accept or reject Ann's proposal. If he rejects they both receive \$0. If on the other hand she offers \$0 to Bob, then he can accept or reject or counteroffer. In the first two cases, the game proceeds identically as with
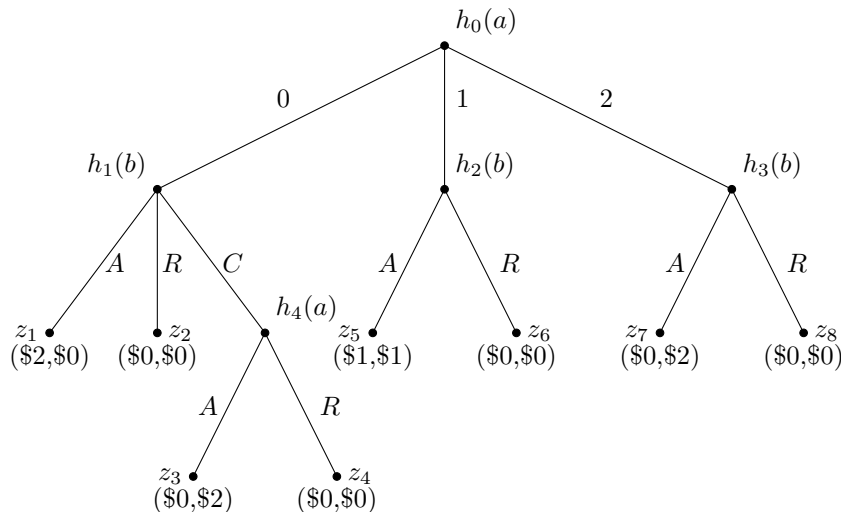
other two offers, whereas in case of a counteroffer, the choice goes back to Ann who either accepts to give all the money to Bob or rejects and they both get nothing. Both players are selfish.   ◁

Question E.1. *Identify the different elements of an extensive-form game (i.e., nodes, active players, actions, outcomes, preferences) in the game of Example E.1.*

STRATEGIES: a strategy is a complete plan of action that specifies what the player does at every node that she is active. Formally, a strategy s_i assigns an action $s_i(h) \in A_i(h)$ to each $h \in H_i$. The set of all strategies is denoted by $S_i := \times_{h \in H_i} A_i(h)$. A strategy is typically assumed to be devised by the player at the beginning of the game, i.e., before the first move at h_0 takes place. Oftentimes a strategy s_i prescribes an action to some node $h \in H_i$ that rules out a node $h' \in H_i$. Nevertheless, s_i still prescribes an action to h'. This is done so that player i is prepared to act following possible mistakes that she may make herself. The importance of being prepared for some mistakes will become clear later on.

Question E.2. *Which are the strategies of each of the two players in the game of Example E.1?*

NODES CONSISTENT WITH A STRATEGY: For a strategy $s_i \in S_i$, the non-terminal nodes that can be reached are $H(s_i)$. By $H_i(s_i) := H(s_i) \cap H_i$ we denote the ones where i is active and can be reached by s_i.

Question E.3. *In the game of Example E.1, for each strategy $s_i \in S_i$ of each player $i \in I$ find the sets $H(s_i)$ and $H_i(s_i)$.*

Question E.4. *Explain the following statement: Each strategy profile uniquely identifies a terminal node. Hence, from the preferences over the set of terminal nodes, we obtain preferences over the set of strategy profiles, $S_a \times S_b$. However, for each terminal node we cannot necessarily identify which strategy profile has been played. Illustrate your answer in the game of Example E.1.*

INFORMATION: we distinguish extensive-form games on the basis of what players observe about past actions that have already taken place. There are two classes of games we consider:

- PERFECT INFORMATION GAMES: The active player at each node knows which are the nodes that have been previously visited, and can therefore deduce what exactly has been previously played.

- IMPERFECT INFORMATION GAMES: The active player at some node does not know which are the nodes that have been previously visited. The interpretation is that some of the past actions have not been observed by the player.

For imperfect information games, we will later on slightly refine the notion of a strategy.

SOLUTION CONCEPT: Similarly to strategic-form games, it is a blackbox that takes as input the structure of the game (i.e., the tuple $\langle I, H, Z, p, (\succeq_i)_{i \in I} \rangle$ for perfect information games) and returns as output either a collection of strategy profiles (i.e., a subset of $S_a \times S_b$). The solution concepts that we will study are classified as follows (for perfect information games):

	ELIMINATION	EQUILIBRIUM
BEGINNING OF THE GAME	–	Nash Equilibrium
THROUGHOUT THE WHOLE GAME	Backward Induction	Subgame Perfect Equilibrium

The row classification distinguishes solution concepts on the basis of whether each player revisits the rationality of her strategies at every node where she is active. The column classification is the same as in the case of strategic-form games. For imperfect information games, we will focus only on Nash Equilibrium and Subgame Perfect Equilibrium.

Question E.5. *Sometimes we only consider the outcomes (i.e., the terminal nodes) that the solution concept predicts rather than the strategy profiles. This is because, it is difficult to obtain data on the strategies that players choose. What does this mean? What is the strategy method in a lab experiment?*

E.1 Perfect information games

CORRESPONDING STRATEGIC FORM: This is the (ordinal) game $\langle I, (S_i)_{i\in I}, (\succeq_i)_{i\in I}\rangle$ that is obtained by describing the strategies of the respective players after having removed all information about the order according to which players move. In this sense, it provides a decent but yet incomplete picture of the game as it does not take the dynamic structure of the game seriously.

Question E.6. *Write the corresponding strategic form of the game in Example E.1.*

Question E.7. *Consider the following game:*
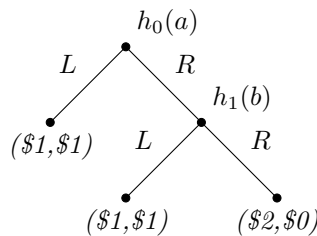
Write the corresponding strategic form. Then, provide an example of a different dynamic game with the same strategic form.

NASH EQUILIBRIUM: This is the Nash Equilibrium of the corresponding strategic form of the game. The idea is that players simultaneously choose a strategy each at the beginning of the game in a way such that they best respond to each other. But at the same time all the analysis is carried out in the corresponding strategic form, meaning that the dynamic structure of the game is irrelevant.

Question E.8. *Find the Nash Equilibria of the game in Example E.1. Are there Nash Equilibria that do not make intuitive sense? Explain.*

Question E.9. *Find the Nash Equilibria in the two games of Question E.7 (i.e., the original game and the one that you constructed with the same strategic form). Are the two sets of Nash Equilibria the same? Is this surprising?*

Question E.10. *Consider the following very well-known game:*
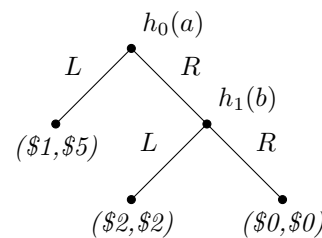
Chain-store paradox

Write the corresponding strategic form and find the Nash Equilibria. Are there Nash Equilibria that do not make intuitive sense?

CORRESPONDING STRATEGIC FORM AT ARBITRARY NODE: Once a node $h \in H$ has been reached, it becomes common knowledge that certain strategies have not been chosen. The strategies of player i that are consistent with h being reached are denoted by $S_i(h)$. Then, the corresponding strategic form of the game at some $h \in H$ is the reduced game $\langle I, (S_i(h))_{i \in I}, (\succeq_i)_{i \in I} \rangle$.

Problem E.1. *Explain the following statement: If h is a direct predecessor of h' then $S_i(h') \subseteq S_i(h)$ for all $i \in I$, with equality holding if $p(h) \neq i$. Illustrate this statement in the context of Example E.1.*

BACKWARD INDUCTION (BI): This is the most common solution concept for extensive form games, and it is perhaps the most intuitive solution concept in game theory. As opposed to Nash Equilibrium, it takes the dynamic structure of the game seriously, but making sure that the players are happy with the strategy they have chosen throughout the entire game, including at nodes with at first sight are irrelevant. Formally, at each step k, Backward Induction eliminates strategies at each node $h \in H_i$ and for every $i \in I$:

$$
\begin{aligned}
S_i^0(h) &:= S_i(h) \\
S_i^1(h) &:= \{s_i \in S_i^0(h) \quad : s_i \text{ is not strictly dominated in } S_i^0(h') \times S_j^0(h') \\
&\qquad\qquad\qquad \text{for any } h' \in H_i(s_i) \text{ that weakly succeeds } h\} \\
S_i^2(h) &:= \{s_i \in S_i^1(h) \quad : s_i \text{ is not strictly dominated in } S_i^1(h') \times S_j^1(h') \\
&\qquad\qquad\qquad \text{for any } h' \in H_i(s_i) \text{ that weakly succeeds } h\} \\
&\;\;\vdots \\
S_i^k(h) &:= \{s_i \in S_i^{k-1}(h) : s_i \text{ is not strictly dominated in } S_i^{k-1}(h') \times S_j^{k-1}(h') \\
&\qquad\qquad\qquad \text{for any } h' \in H_i(s_i) \text{ that weakly succeeds } h\} \\
&\;\;\vdots
\end{aligned}
$$

Then, the strategies of player i that survive BI are those in $S_i^\infty := \bigcap_{k=0}^\infty S_i^k(h_0)$. The predictions of BI are the strategy profiles in $BI := S_a^\infty \times S_b^\infty$. The BI algorithm described above is equivalent to the following procedure, which can be directly applied on the extensive form of the game. The nodes that are followed only by terminal nodes are called Nodes of Group 1. The nodes that are direct predecessors to Nodes of Group 1 are called Nodes of Group 2, and inductively the nodes that are direct predecessors to Nodes of Group k are called Nodes of Group $k+1$. Then, begin by eliminating the actions that are not optimal at the Nodes of Group 1. Then, eliminate the actions that are never optimal at the Nodes of Group 2, given that the Nodes of Group 1 have been deleted. Continue inductively. In the end take the strategy profiles that use only the remaining actions.

Question E.11. *Apply Backward Induction in the game of Example E.1 as well as those of Questions E.7 and E.10. Compare the predictions of BI with those of NE. In the game of Question E.7 compare the predictions of BI in the original game with those of BI in the game that you constructed with the same strategic form.*

Question E.12. *Explain why the two BI procedures that we describe above are equivalent. Illustrate your answer in the context of Example E.1.*

Theorem E.1. *If the game has no ties, there is a unique strategy profile surviving BI.*

37

Question E.13. *In the context of Example E.1 illustrate that $BI \not\subseteq NE$ and $NE \not\subseteq BI$. Provide intuition. Nevertheless, if the game does not have ties, it will be the case that $BI \subseteq NE$. Illustrate your answer in the context of Question E.10.*

Question E.14. *Consider the following very well-known game:*

Centipede game
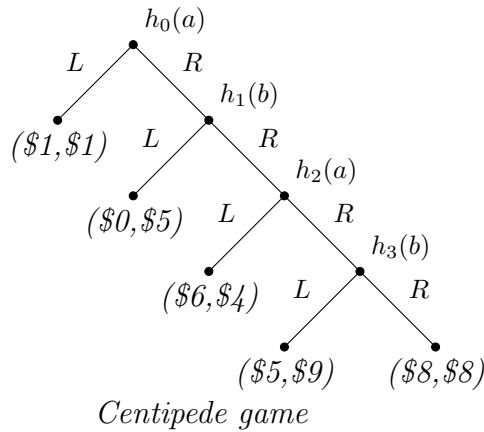
What does BI predict? What do you think most subjects do? Provide an explanation.

CONTINUATION STRATEGIES: Here it will become clear why strategies prescribe actions also at nodes that are precluded by previous own actions. For a strategy $s_i \in S_i$, define the strategy $s_i^h \in S_i(h)$ as follows: at all nodes h' that precede h take $s_i^h(h')$ to be the action that leads from h' towards h, while at all other nodes h'' take $s_i^h(h'') = s_i(h'')$. Of course, if $h \in H(s_i)$ then $s_i^h = s_i$.

Question E.15. *In the game of Example E.1 take the strategy $s_a := 1R$. Which are the continuation strategies at every node where Ann is active?*

SUBGAME PERFECT EQUILIBRIUM (SPE): It is a strategy profile which is a Nash Equilibrium at every $h \in H$. Formally, (s_a, s_b) is a Subgame Perfect Equilibrium, if (s_a^h, s_b^h) is a Nash Equilibrium in the game $S_a(h) \times S_b(h)$ for every $h \in H$. In other words, a Subgame Perfect Equilibrium is a Nash Equilibrium which is robust to mistakes that players may make in the implementation of their strategies. The set of Subgame Perfect Equilibria is denoted by $SPE \subseteq S_a \times S_b$. In perfect information games, at least one SPE always exists.

Question E.16. *Compute the SPE in the games of Example E.1 and Questions E.7 and E.10.*

Theorem E.2. *A Subgame Perfect Equilibrium survives Backward Induction.*

Problem E.2. *Not all Nash Equilibria that survive Backward Induction are necessarily Subgame Perfect Equilibria. Illustrate the previous statement in the following example:*
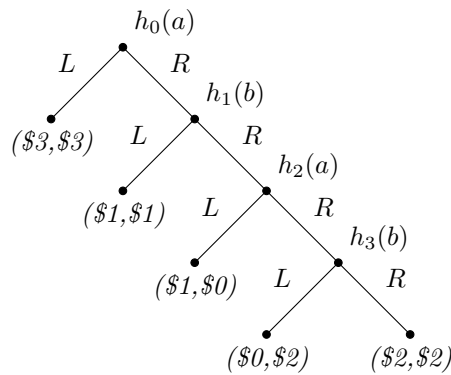
In particular, find a Nash Equilibrium which survives Backward Induction, but is not a Subgame Perfect Equilibrium.

Question E.17. *Prove the following statement: In perfect information games without ties, the Subgame Perfect Equilibrium gives a unique prediction.*

Question E.18. *Consider the following well-known games:*

- ULTIMATUM GAME*: Ann proposes a split of \$10 into integer amounts between herself and Bob. If Bob accepts, the proposed allocation is implemented. If he rejects, they both receive \$0.*

- TRUST GAME: *Ann chooses a split of \$10 into integer amounts between herself and Bob. The amount that Bob receives is doubled. Then, Bob decided how to split his (doubled) amount into integer amounts between himself and Ann.*

In the previous two games, find the Nash Equilibria and the Subgame Perfect Equilibria.

E.2 Imperfect information games

INFORMATION SETS: We often take some non-terminal nodes that are controlled by i and we bundle them together. Intuitively, $I_i \subseteq H_i$ is an information set whenever the following holds: every time time some $h \in I_i$ is reached, player i does not know that h has been indeed reached, and instead considers possible all the nodes in I_i. As a consequence, if $h, h' \in I_i$ then $A_i(h) = A_i(h')$. Formally, $\mathcal{I}_i$ is the collection of all information sets of player i.

Example E.2. Consider the following variant of the game from Example E.1: everything remains the same as before, except for the fact that Bob does not observe
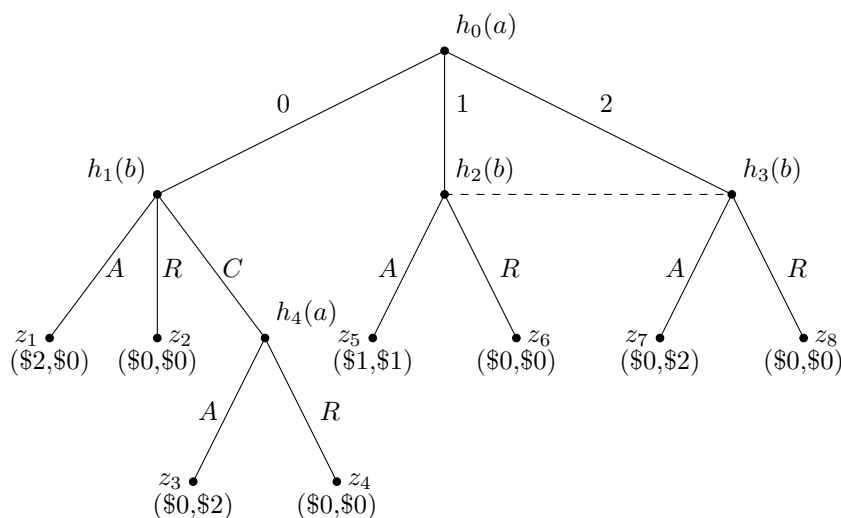
whether Ann has chosen 1 or 2 at h_0. Bob's failure to observe Ann's past action in these cases is modelled by bundling together h_2 and h_3 into one information set. This is graphically illustrated by connecting these two nodes with a dashed line. ◁

Question E.19. *Which are the information sets in the perfect information game of Example E.1?*

Question E.20. *Explain the following statement: whenever $h, h' \in I_i$ it must necessarily be the case that i chooses the same action at h and at h'.*

STRATEGIES: a strategy is still a complete plan of action that specifies what the player does at every node that she is active, subject to the restriction discussed in Question E.20. Formally, a strategy s_i assigns an action $s_i(h) \in A_i(h)$ to each $h \in H_i$, such that $s_i(h) = s_i(h')$ for all $h, h' \in I_i$ and all $I_i \in \mathcal{I}_i$. In this sense, s_i can be rewritten as a function that assigns an action $s_i(I_i)$ to each information set $I_i \in \mathcal{I}_i$. The set of all strategies is thus denoted by $S_i := \times_{I_i \in \mathcal{I}_i} A_i(I_i)$.

Question E.21. *Which are the strategies of each of the two players in the game of Example E.2?*

KNOWLEDGE: Each information set I_i is identified by a subset of $S_a \times S_b$, i.e., by the strategy profiles that are consistent with reaching this information set. The same is true for every other relevant event in the game, e.g., player i choosing action $a_i \in A_i(h)$ is identified by the collection of all strategy profiles that prescribe a_i to h. Then, we say that an event E is known at the information set I_i, if the collection of profiles that identifies I_i is a subset of the collection of profiles that identifies the event E.

Problem E.3. *In the game of Example E.2, identify by means of a subset of $S_a \times S_b$ Bob's information set $\{h_2, h_3\}$. Similarly, identify the event that Ann has picked the action $R \in A_a(h_4)$.*

  (a) Does Bob know at $\{h_2, h_3\}$ that Ann has picked R at h_4?

  (b) Does Bob know at $\{h_2, h_3\}$ what Ann has played in the past? Explain using the formal way of representing knowledge.

PERFECT RECALL: Each player remembers what she did in the past and what she knew in the past. Henceforth, we assume that the information sets satisfy the following conditions:

- RECALL OF PAST OWN ACTIONS: if h precedes h', and the corresponding information sets are $h \in I_i$ and $h' \in I_i$, then player i knows at h' which action she took herself at h.

- RECALL OF PAST OWN KNOWLEDGE: if h precedes h', and the corresponding information sets are $h \in I_i$ and $h' \in I_i$, then player i knows at h' every event she knew at h'.

Question E.22. *Suppose that player i is absentminded, i.e., there are two succeeding nodes $h, h' \in I_i$. Provide an example with an absentminded player, and show that recall of past actions is violated.*

Question E.23. *Provide an example where recall of past knowledge is violated.*

SUBGAME: It a reduced extensive-form game (obtained by deleting parts of the tree) which satisfies the following conditions: (a) it has a unique root $h \in H$, (b) if $h \in I_i$ belongs to the subgame, so does every other $h' \in I_i$, (c) if h belongs to the subgame, so does every succeeding (non-terminal or terminal) node. Every subgame is identified by its root. Obviously, in a perfect information game, every non-terminal node is the root of a subgame.

**Question E.24.** *Which are the subgames in Example E.2?*

CORRESPONDING STRATEGIC FORM AT A SUBGAME: The root h of a subgame has been reached, take the strategies of each player i that are consistent with h and define them by $S_i(h)$. Then, similarly to the perfect information case, the corresponding strategic form of the game at this subgame becomes $S_i(h) \times S_j(h)$.

NASH EQUILIBRIUM: Identically to perfect information games, it is the Nash Equilibrium of the corresponding normal form at h_0.

SUBGAME PERFECT EQUILIBRIUM: It is a strategy profile which is a Nash Equilibrium at every subgame. Formally, (s_a, s_b) is a Subgame Perfect Equilibrium, if (s_a^h, s_b^h) is a Nash Equilibrium in the game $S_a(h) \times S_b(h)$ for every node h that identifies some subgame, where the continuation strategy s_i^h is defined identically to the perfect information case. The set of Subgame Perfect Equilibria is again denoted by $SPE \subseteq S_a \times S_b$.

Question E.25. *Find the Nash Equilibria and the Subgame Perfect Equilibria in Example E.2.*

Problem E.4. *In imperfect information games, if we only know the ordinal utilities, we cannot always find a Subgame Perfect Equilibrium. Illustrate the previous statement in the following example:*
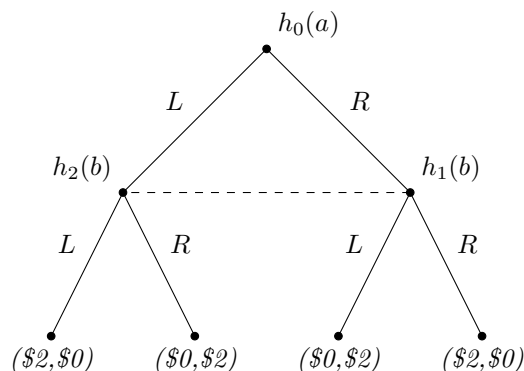
In particular, show that this game does not have any Subgame Perfect Equilibrium.

Part III
Mathematical Appendix

F   Set theory

We start with a grand set X. For a typical element of X is denoted by x. A subset of X is a collection of some (perhaps all) of the elements of X. Whenever an element x belongs to a subset A, we write $x \in A$. Otherwise, we write $x \notin A$. The empty set is denoted by $\emptyset$ and it does not contain any element.

INCLUSION: We say that A is a subset of B, and we write $A \subseteq B$, whenever $x \in A$ implies $x \in B$. Obviously, $\emptyset \subseteq A \subseteq X$ for every subset A.

UNION: $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

INTERSECTION: $x \in A \cap B$ if and only if $x \in A$ and $x \in B$. We say that A and B are disjoint whenever $A \cap B = \emptyset$.

COMPLEMENT: A and B are complements whenever $A \cup B = X$ and $A \cap B = \emptyset$.

SET DIFFERENCE: $x \in A \setminus B$ if and only if $x \in A$ and $x \notin B$.

G   Calculus

A function $f : X \to Y$ is a mapping that takes each element $x \in X$ and associates it with an element $f(x) \in Y$.

STRICTLY INCREASING FUNCTION: a function $f : \mathbb{R} \to \mathbb{R}$ such that $x > y$ implies $f(x) > f(y)$.

LINEAR FUNCTION: a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \alpha + \beta x$ for $\alpha, \beta \in \mathbb{R}$. Whenever $\beta > 0$, it is obviously strictly increasing.

TRANSFORMATION: take a function $u : \mathbb{R} \to \mathbb{R}$, and another function $f : \mathbb{R} \to \mathbb{R}$. Then, the function $f(u)$ is a transformation of u, and it assigns to each $x \in \mathbb{R}$ value $f(u(x))$.

H   Probability theory

We start with a (finite) state space, which is a set X. Each element $x \in X$ is called a state, and it is interpreted as a full description of everything that is relevant, i.e., once we know which state is the true one, all relevant uncertainty has been resolved. A subset $A \subseteq X$ is called an event.

RANDOM VARIABLE: it is a real-valued function $u : X \to \mathbb{R}$ that specifies a value to an object of interest at each state.

PROBABILITY DISTRIBUTION: it assigns to each $x \in X$ a probability $\mu(x) \in [0, 1]$ in a way such that $\sum_{x \in X} \mu(x) = 1$. The probability of an event $A \subseteq X$ is then equal to $\sum_{x \in A} \mu(x)$. The probability of the empty set is 0.

EXPECTED VALUE: for a random variable u and probability distribution μ, the expected value is

$$\mathbb{E}_\mu(u) = \sum_{x \in X} \mu(x)u(x).$$